Lecture 2: Orbital dynamics

2.1. Orbits in an axisymmetric potential

Consider the motion, according to Newtonian dynamics, of a test particle in the gravitational potential Φ of a star, planet, black hole, galaxy, etc. Use cylindrical polar coordinates (r, ϕ, z) (called *radial*, *azimuthal* and *vertical*).

Assume that Φ is *axisymmetric* and *reflectionally symmetric*:

$$\Phi = \Phi(r, z), \qquad \Phi(r, -z) = \Phi(r, z).$$

An important special case is the potential of a point mass M,

$$\Phi = -\frac{GM}{\sqrt{r^2 + z^2}}.$$

In this case the test particle follows a Keplerian orbit.

Lagrange's equations of motion are

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} = \frac{\partial L}{\partial q_i},$$

where q_i are the generalized coordinates of the particle. The Lagrangian for a particle of unit mass is

$$L = T - V = \frac{1}{2}\left(\dot{r}^2 + r^2\dot{\phi}^2 + \dot{z}^2\right) - \Phi(r, z).$$

Two conserved quantities are the *specific angular momentum*,

$$h = \frac{\partial L}{\partial \dot{\phi}} = r^2\dot{\phi},$$

and the *specific energy*,

$$\varepsilon = \sum_i \dot{q}_i \frac{\partial L}{\partial \dot{q}_i} - L = T + V = \frac{1}{2}\left(\dot{r}^2 + r^2\dot{\phi}^2 + \dot{z}^2\right) + \Phi(r, z).$$

The radial and vertical equations of motion are

$$\ddot{r} = r\dot{\phi}^2 - \Phi_r, \qquad \ddot{z} = -\Phi_z,$$

where the subscripts on Φ denote partial derivatives. These are equivalent to

$$\ddot{r} = -\Phi^{\text{eff}}_r, \qquad \ddot{z} = -\Phi^{\text{eff}}_z,$$

with the *effective potential*

$$\Phi^{\text{eff}} = \frac{h^2}{2r^2} + \Phi.$$

Note that

$$\varepsilon = \frac{1}{2}\left(\dot{r}^2 + \dot{z}^2\right) + \Phi^{\text{eff}}.$$

Consider the family of circular orbits ($r = \text{constant}$) in the midplane ($z = 0$). These must satisfy

$$
\begin{aligned}
0 &= -\Phi^{\text{eff}}_r(r,0) = \frac{h^2}{r^3} - \Phi_r(r,0), \\
0 &= -\Phi^{\text{eff}}_z(r,0) \qquad (\checkmark \text{ by reflectional symmetry}).
\end{aligned}
$$

The specific angular momentum $h_\text{c}(r)$, angular velocity $\Omega_\text{c}(r)$ and specific energy $\varepsilon_\text{c}(r)$ of the circular orbits are therefore given by (assuming $\Phi_r(r,0) > 0$ and considering prograde orbits)

$$
h_\text{c} = \sqrt{r^3 \Phi_r(r,0)}, \qquad \Omega_\text{c} = \frac{h_\text{c}}{r^2}, \qquad \varepsilon_\text{c} = \frac{h_\text{c}^2}{2r^2} + \Phi(r,0).
$$

They satisfy the relation

$$
\frac{d\varepsilon_\text{c}}{dh_\text{c}} = \Omega_\text{c}.
$$

Proof:

$$
\frac{d\varepsilon_\text{c}}{dr} = \frac{h_\text{c}}{r^2}\frac{dh_\text{c}}{dr} - \frac{h_\text{c}^2}{r^3} + \Phi_r(r,0) = \Omega_\text{c}\frac{dh_\text{c}}{dr}.
$$

The *orbital shear rate* $S(r)$ and the dimensionless *orbital shear parameter* $q(r)$ are defined by

$$
S = -r\frac{d\Omega_\text{c}}{dr}, \qquad q = -\frac{d\ln\Omega_\text{c}}{d\ln r} = \frac{S}{\Omega_\text{c}}.
$$

In the case of a point-mass potential, the circular Keplerian orbits satisfy

$$
\Phi(r,0) = -\frac{GM}{r}, \qquad h_\text{c} = \sqrt{GMr}, \qquad \Omega_\text{c} = \sqrt{\frac{GM}{r^3}}, \qquad \varepsilon_\text{c} = -\frac{GM}{2r}, \qquad S = \frac{3}{2}\Omega_\text{c}, \qquad q = \frac{3}{2}.
$$

(See Example 1.1 for a revision of Keplerian orbits.)

2.2. Oscillations and precession

Small departures from a circular orbit of radius r in the midplane satisfy

$$
\ddot{\delta r} = -\Omega_r^2\,\delta r, \qquad \ddot{\delta z} = -\Omega_z^2\,\delta z,
$$

with

$$
\Omega_r^2 = \Phi^{\text{eff}}_{rr}(r,0), \qquad \Omega_z^2 = \Phi^{\text{eff}}_{zz}(r,0),
$$

defining the *radial frequency* $\Omega_r(r)$ and the *vertical frequency* $\Omega_z(r)$. (The radial frequency is more often called the *epicyclic frequency* and denoted κ. The vertical frequency is sometimes denoted ν. Note that $\Phi^{\text{eff}}_{rz}(r,0) = 0$ by reflectional symmetry.)

The circular orbit is stable if $\Omega_r^2 > 0$ and $\Omega_z^2 > 0$, i.e. if the orbit minimizes ε for a given h.

We have

$$
\begin{aligned}
\Omega_r^2 &= \frac{3h_\text{c}^2}{r^4} + \Phi_{rr}(r,0) \\
&= \frac{3h_\text{c}^2}{r^4} + \frac{d}{dr}\left(\frac{h_\text{c}^2}{r^3}\right) \\
&= \frac{1}{r^3}\frac{dh_\text{c}^2}{dr} \\
&= 4\Omega_\text{c}^2 + 2r\Omega_\text{c}\frac{d\Omega_\text{c}}{dr} \\
&= 2\Omega_\text{c}(2\Omega_\text{c} - S) \\
&= 2(2-q)\Omega_\text{c}^2, \\
\Omega_z^2 &= \Phi_{zz}(r,0).
\end{aligned}
$$

Keplerian orbits satisfy

$$\Omega_r = \Omega_z = \Omega,$$

meaning that (slightly) eccentric or inclined orbits close after one turn.

[FIGURE]

If $\Omega_r \approx \Omega$, an eccentric orbit precesses slowly. The minimum radius (*periapsis*) occurs at time intervals $\Delta t = 2\pi/\Omega_r$, corresponding to

$$\begin{aligned}
\Delta\phi &= \frac{2\pi\Omega}{\Omega_r} \\
&= 2\pi\left(\frac{\Omega}{\Omega_r} - 1\right) + 2\pi \\
&= 2\pi\left(\frac{\Omega}{\Omega_r} - 1\right) \bmod 2\pi.
\end{aligned}$$

The *apsidal precession rate* is therefore

$$\frac{\Delta\phi}{\Delta t} = \Omega - \Omega_r.$$

Similarly, if $\Omega_z \approx \Omega$, an inclined orbit precesses slowly with *nodal precession rate*

$$\Omega - \Omega_z.$$

(See Example 1.2 for precession of orbits in binary stars and around black holes.)

2.3. Mechanics of accretion

Consider two particles in circular orbits in the midplane. Can energy be released by a conservative exchange of angular momentum between the particles?

The total angular momentum and energy are

$$\begin{aligned}
H &= H_1 + H_2 = m_1 h_1 + m_2 h_2, \\
E &= E_1 + E_2 = m_1 \varepsilon_1 + m_2 \varepsilon_2.
\end{aligned}$$

In an infinitesimal exchange:

$$\begin{aligned}
dH &= dH_1 + dH_2 = m_1\, dh_1 + m_2\, dh_2, \\
dE &= dE_1 + dE_2 = m_1 \Omega_1\, dh_1 + m_2 \Omega_2\, dh_2,
\end{aligned}$$

If $dH = 0$ then

$$dE = (\Omega_1 - \Omega_2)\, dH_1.$$

So energy is released by transferring angular momentum from higher to lower angular velocity. In practice $d\Omega/dr < 0$, so this means an *outward transfer of angular momentum.*

Now generalize the argument to allow for an exchange of mass:

$$\begin{aligned}
dM &= dm_1 + dm_2 = 0, \\
dH &= dH_1 + dH_2 = 0, \qquad dH_i = m_i\, dh_i + h_i\, dm_i, \\
dE_i &= m_i \Omega_i\, dh_i + \varepsilon_i\, dm_i \\
&= \Omega_i\, dH_i + (\varepsilon_i - h_i \Omega_i)\, dm_i, \\
dE &= (\Omega_1 - \Omega_2)\, dH_1 + [(\varepsilon_1 - h_1 \Omega_1) - (\varepsilon_2 - h_2 \Omega_2)]\, dm_1.
\end{aligned}$$

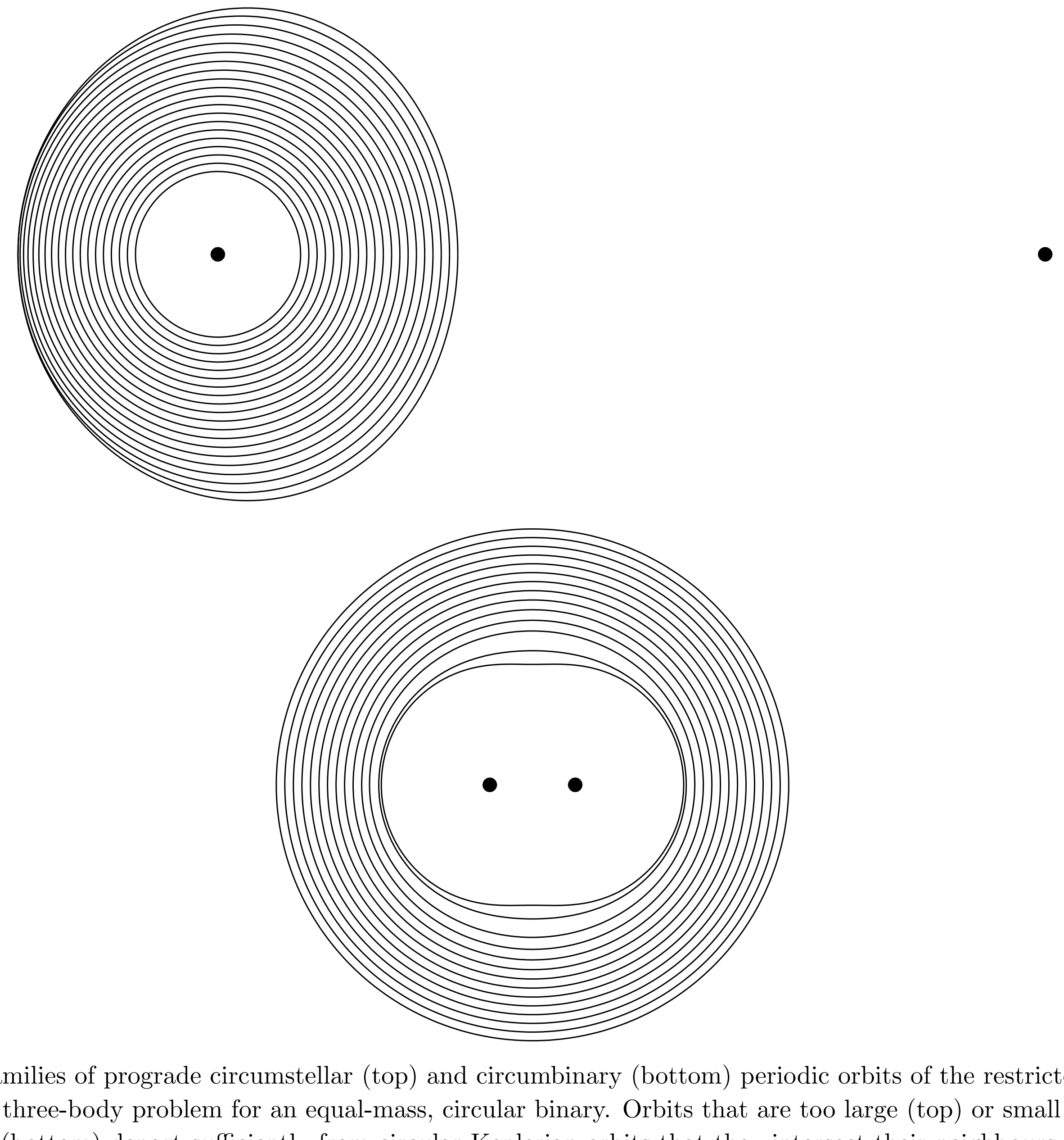
In practice

$$\frac{d}{dr}(\varepsilon - h\Omega) = -h\,\frac{d\Omega}{dr} > 0,$$

so energy is released by an *outward transfer of angular momentum* and an *inward transfer of mass.* This is the physical basis of an accretion disc.

[FIGURE]

2.4. Departures from Keplerian rotation



Families of prograde circumstellar (top) and circumbinary (bottom) periodic orbits of the restricted three-body problem for an equal-mass, circular binary. Orbits that are too large (top) or small (bottom) depart sufficiently from circular Keplerian orbits that they intersect their neighbours.

Exercise: Accretion on to a non-rotating black hole can be modelled using the potential $\Phi = -GM/(R-r_\mathrm{h})$, where $R = \sqrt{r^2+z^2}$ is the spherical radius and $r_\mathrm{h} = 2GM/c^2$ is the (Schwarzschild) radius of the event horizon of the black hole. Calculate $\Omega_\mathrm{c}(r)$ and compare with the Keplerian angular velocity. Show that $h_\mathrm{c}(r)$ has a minimum at $r = 3r_\mathrm{h}$ and deduce that circular orbits in this potential are unstable for $r < 3r_\mathrm{h}$.

Lecture 3: Global and local views

In the *global view* of an astrophysical disc we consider the full disc, usually in cylindrical polar coordinates (r, ϕ, z).

In the *local view* (known as the local approximation, local model, shearing sheet, shearing box, Hill's approximation, etc.) we consider a small volume of the disc in the neighbourhood of an orbiting reference point, using local Cartesian coordinates (x, y, z) in the radial, azimuthal and vertical directions.

3.1. Local view of orbital dynamics

Select a reference point in a circular orbit of radius r_0 in the midplane. The orbit has angular velocity $\Omega_0 = \Omega(r_0)$, etc. (We drop the subscript 'c' on properties of circular orbits, but use a subscript '0' for the time being to indicate evaluation on the reference orbit.)

Introduce local coordinates (x, y, z) through

$$r = r_0 + x, \qquad \phi = \Omega_0 t + \frac{y}{r_0}, \qquad z = z.$$

Expand the Lagrangian for a particle of unit mass,

$$\begin{aligned} L &= \frac{1}{2}\left(\dot{r}^2 + r^2\dot{\phi}^2 + \dot{z}^2\right) - \Phi(r, z) \\ &= \frac{1}{2}\left(\dot{x}^2 + (r_0 + x)^2\left(\Omega_0 + \frac{\dot{y}}{r_0}\right)^2 + \dot{z}^2\right) - \Phi(r_0 + x, z), \end{aligned}$$

to second order in (x, y, z) to obtain

$$L = L_0 + L_1 + L_2 + \dots,$$

where

$$\begin{aligned} L_0 &= \frac{1}{2}r_0^2\Omega_0^2 - \Phi_0 = \text{constant}, \\ L_1 &= r_0\Omega_0\dot{y} + (r_0\Omega_0^2 - \Phi_{r0})x = r_0\Omega_0\dot{y} = \frac{d}{dt}(r_0\Omega_0 y), \\ L_2 &= \frac{1}{2}\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) + 2\Omega_0 x\dot{y} + \frac{1}{2}\Omega_0^2 x^2 - \frac{1}{2}\Phi_{rr0}x^2 - \frac{1}{2}\Phi_{zz0}z^2. \end{aligned}$$

The terms L_0 (a constant) and L_1 (a total time-derivative) make no contribution to Lagrange's equations and generate no motion. For small (x, y, z), the motion is dominated by L_2.

L_2 is separable into horizontal and vertical parts:

$$L_2 = L_\mathrm{h} + L_\mathrm{v} = \left[\frac{1}{2}\left(\dot{x}^2 + \dot{y}^2\right) + 2\Omega_0 x\dot{y} - \frac{1}{2}(\Phi_{rr0} - \Omega_0^2)x^2\right] + \left[\frac{1}{2}\dot{z}^2 - \frac{1}{2}\Phi_{zz0}z^2\right].$$

The motion due to L_2 is

$$\begin{aligned} \ddot{x} &= 2\Omega_0\dot{y} - (\Phi_{rr0} - \Omega_0^2)x, \\ \ddot{y} + 2\Omega_0\dot{x} &= 0, \\ \ddot{z} &= -\Phi_{zz0}z. \end{aligned}$$

At this level of approximation, (x, y, z) may be interpreted as Cartesian coordinates in a frame rotating with the orbital angular velocity Ω_0. The equation of motion includes the Coriolis force and the *tidal potential*

$$\Phi_{\mathrm{t}} = \frac{1}{2}(\Phi_{rr0} - \Omega_0^2)x^2 + \frac{1}{2}\Phi_{zz0}z^2,$$

which comes from the expansion of the sum of the gravitational and centrifugal potentials, $\Phi - \frac{1}{2}\Omega_0^2 r^2$, to second order in (x, z).

From the radial force balance,

$$\Phi_r(r, 0) = r\Omega^2 \qquad \Rightarrow \qquad \Phi_{rr}(r, 0) = \Omega^2 - 2\Omega S,$$

so the tidal potential is

$$\Phi_{\mathrm{t}} = -\Omega_0 S_0 x^2 + \frac{1}{2}\Omega_{z0}^2 z^2.$$

In practice the orbital shear $S > 0$, so Φ_{t} has a saddle point at the origin.

Rewrite the equations of motion in the local view as

$$\begin{aligned}
\ddot{x} - 2\Omega_0\dot{y} &= 2\Omega_0 S_0 x, \\
\ddot{y} + 2\Omega_0\dot{x} &= 0, \\
\ddot{z} &= -\Omega_{z0}^2 z.
\end{aligned}$$

Three conserved quantities are

$$\begin{aligned}
p_y &= \frac{\partial L_2}{\partial \dot{y}} = \dot{y} + 2\Omega_0 x, \\
\varepsilon_{\mathrm{h}} &= \sum_i \dot{q}_i \frac{\partial L_{\mathrm{h}}}{\partial \dot{q}_i} - L_{\mathrm{h}} = \frac{1}{2}\left(\dot{x}^2 + \dot{y}^2\right) - \Omega_0 S_0 x^2, \\
\varepsilon_{\mathrm{v}} &= \sum_i \dot{q}_i \frac{\partial L_{\mathrm{v}}}{\partial \dot{q}_i} - L_{\mathrm{v}} = \frac{1}{2}\dot{z}^2 + \frac{1}{2}\Omega_{z0}^2 z^2.
\end{aligned}$$

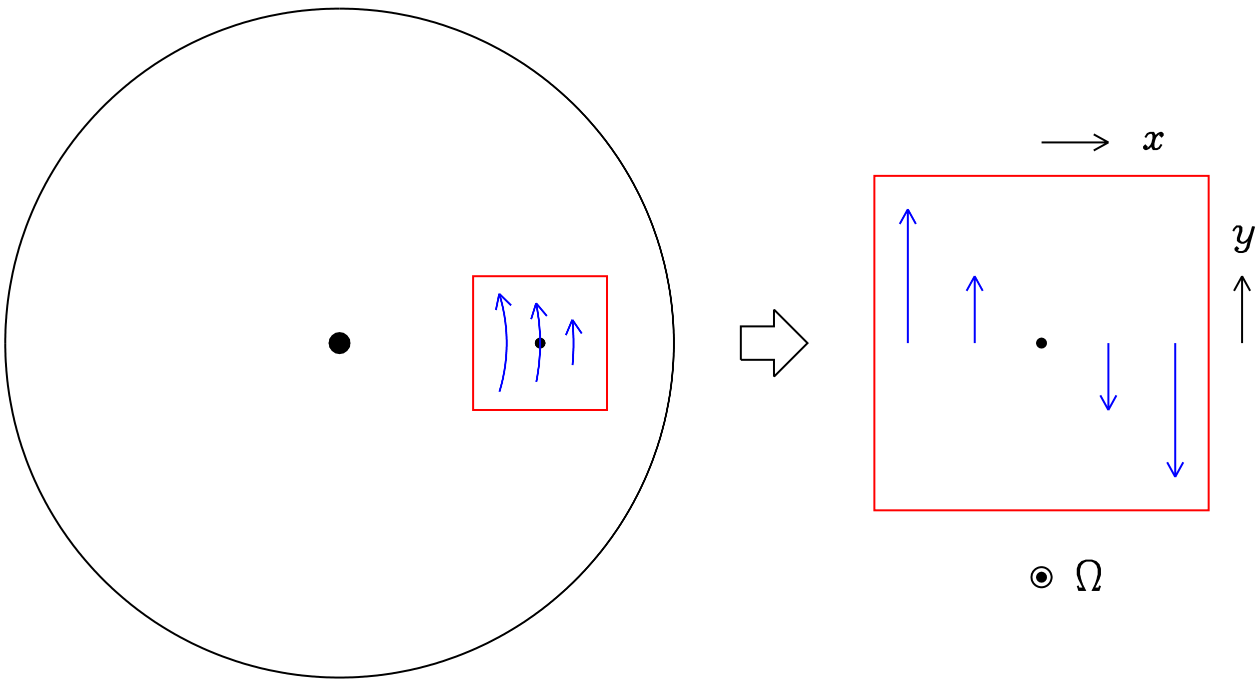
These can be related to the expansions of h and ε in the local view. Consider the conserved quantities

$$\begin{aligned}
\frac{h}{r_0} &= \frac{r^2\dot{\phi}}{r_0} = \frac{(r_0 + x)^2}{r_0}\left(\Omega_0 + \frac{\dot{y}}{r_0}\right) = \text{constant} + (\dot{y} + 2\Omega_0 x) + \cdots, \\
\varepsilon - \Omega_0 h &= \frac{1}{2}\left(\dot{r}^2 + r^2\dot{\phi}^2 + \dot{z}^2\right) + \Phi(r, z) - \Omega_0 r^2\dot{\phi} \\
&= \frac{1}{2}\left(\dot{x}^2 + (r_0 + x)^2\left(\Omega_0 + \frac{\dot{y}}{r_0}\right)^2 + \dot{z}^2\right) + \Phi(r_0 + x, z) - \Omega_0(r_0 + x)^2\left(\Omega_0 + \frac{\dot{y}}{r_0}\right) \\
&= \text{constant} + \frac{1}{2}\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) - \Omega_0 S_0 x^2 + \frac{1}{2}\Omega_{z0}^2 z^2 + \cdots \qquad (\textbf{exercise}).
\end{aligned}$$

The local representation of the family of circular orbits in the midplane is

$$x = \text{constant}, \qquad \dot{y} = -S_0 x, \qquad z = 0,$$

which can be interpreted as an *orbital shear flow* with shear rate $S_0 = q_0\Omega_0$. Note that, in the rotating frame of the local view, the Coriolis force balances the tidal force for these orbits.



To obtain the general solution of the local equations of motion, note that

$$\begin{aligned}
&\ddot{x} - 2\Omega_0(p_y - 2\Omega_0 x) = 2\Omega_0 S_0 x\\
&\ddot{x} + 2\Omega_0(2\Omega_0 - S_0)x = 2\Omega_0 p_y\\
&\ddot{x} + \Omega_{r0}^2 x = 2\Omega_0 p_y,
\end{aligned}$$

so

$$\begin{aligned}
x &= x_0 + \mathrm{Re}\left(A\, e^{-i\Omega_{r0}t}\right),\\
y &= y_0 - S_0 x_0 t + \mathrm{Re}\left(\frac{2\Omega_0 A}{i\Omega_{r0}}\, e^{-i\Omega_{r0}t}\right),\\
z &= \mathrm{Re}\left(B\, e^{-i\Omega_{z0}t}\right),
\end{aligned}$$

for some real constant x_0 and complex oscillation amplitudes A and B. These are the local representation of (slightly) eccentric and inclined orbits. The three conserved quantities evaluate to (**exercise**)

$$\begin{aligned}
p_y &= (2\Omega_0 - S_0)x_0,\\
\varepsilon_{\mathrm{h}} &= \frac{1}{2}\Omega_{r0}^2\left(|A|^2 - \frac{S_0}{2\Omega_0}x_0^2\right),\\
\varepsilon_{\mathrm{v}} &= \frac{1}{2}\Omega_{z0}^2|B|^2.
\end{aligned}$$

Having derived the local model, we usually omit the subscript zero on Ω, S, Ω_r, Ω_z, etc. These quantities are regarded as constants, evaluated at r_0.

3.2. Symmetries of the local model

The local view has some symmetries inherited from the global view:

- Translational symmetry in y: $y \mapsto y + c$
- Reflectional symmetry in z: $z \mapsto -z$

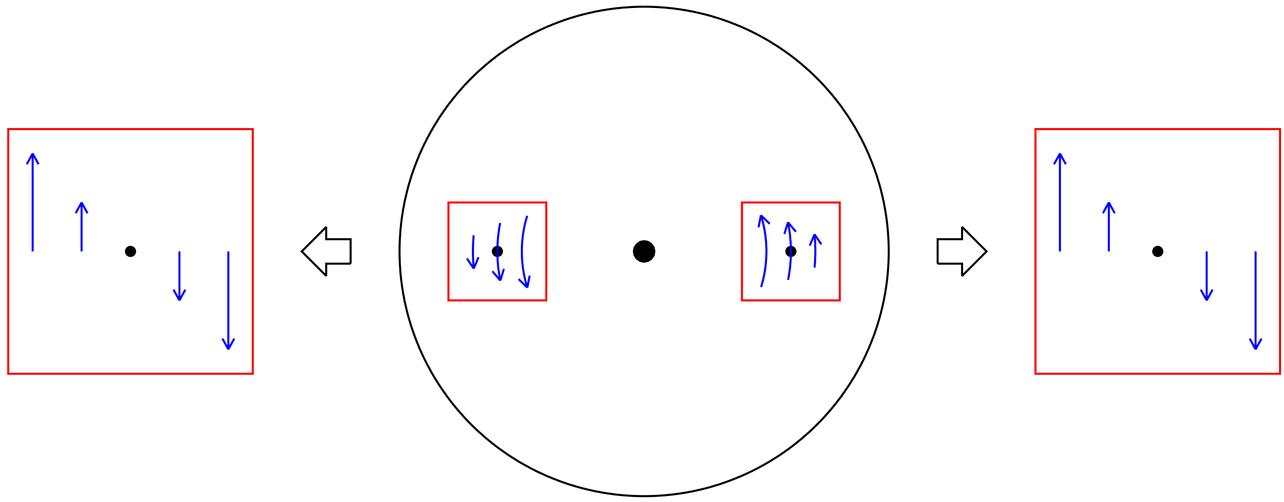
It has additional symmetries not present in the global view:

- Translational symmetry in x (when combined with a Galilean boost in y):

$$x \mapsto x + c, \qquad y \mapsto y - S_0 ct$$

- Rotational symmetry by π about the z axis (we cannot tell the inside from the outside):

$$x \mapsto -x, \qquad y \mapsto -y$$



- Scale-invariance (no characteristic length-scale, because we zoomed in to scales $\ll r_0$):

$$\mathbf{x} \mapsto c\,\mathbf{x}$$

- Separability of horizontal and vertical dimensions

The combination of translational symmetries in x and y means that the local model is *horizontally homogeneous.*

Exercise: Interpret the motion of a test particle in the local approximation for a Keplerian disc ($\Omega_r = \Omega_z = \Omega$). Starting with the case in which $x_0 = 0$ and $B = 0$ but $A \neq 0$, show that the epicyclic motion consists of a retrograde ellipse with an axis ratio of 2. Show that including $B \neq 0$ produces a tilted ellipse, and that including $x_0 \neq 0$ makes the centre of the ellipse (the *guiding centre* of the epicycle) drift in the azimuthal direction at the orbital shear rate.

Lecture 4: Evolution of an accretion disc

4.1. Conservation of mass and angular momentum

The evolution of an accretion disc is regulated by the conservation of mass and angular momentum. These are embodied in the 3D equations of fluid dynamics, which we reduce to a 1D form by integration.

The equation of mass conservation is

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \cdot (\rho \mathbf{u}) = 0,$$

where ρ is the mass density and $\mathbf{u}$ is the velocity. In cylindrical polar coordinates,

$$\frac{\partial \rho}{\partial t} + \frac{1}{r}\frac{\partial}{\partial r}(r\rho u_r) + \frac{1}{r}\frac{\partial}{\partial \phi}(\rho u_\phi) + \frac{\partial}{\partial z}(\rho u_z) = 0.$$

Integrate this equation over the cylinder $\mathscr{C}_r$ of radius r:

$$\int_{\mathscr{C}_r} \cdot \, dA = \int_{-\infty}^{\infty} \int_0^{2\pi} \cdot \, r \, d\phi \, dz.$$

Assuming no loss or gain through the vertical boundaries, we obtain

$$\frac{\partial \mathscr{M}}{\partial t} + \frac{\partial \mathscr{F}}{\partial r} = 0,$$

where

$$\mathscr{M}(r,t) = \int_{\mathscr{C}_r} \rho \, dA$$

is the 1D mass density (mass per unit radius) and

$$\mathscr{F}(r,t) = \int_{\mathscr{C}_r} \rho u_r \, dA$$

is the radial mass flux. Accretion corresponds to radial inflow ($\mathscr{F} < 0$).

The equation of angular-momentum conservation comes from the equation of motion, which we write in the form

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \boldsymbol{\nabla} \mathbf{u} = -\boldsymbol{\nabla}\Phi + \frac{1}{\rho}\boldsymbol{\nabla} \cdot \mathsf{T},$$

where the symmetric stress tensor T accounts for momentum transport due to the collective effects of the fluid (pressure, viscosity, magnetic fields, self-gravity, turbulence, etc.). Combine this with the equation of mass conservation to obtain the equation of momentum conservation,

$$\frac{\partial}{\partial t}(\rho \mathbf{u}) + \boldsymbol{\nabla} \cdot (\rho \mathbf{u}\mathbf{u} - \mathsf{T}) = -\rho \boldsymbol{\nabla}\Phi.$$

Here Φ is the external gravitational potential in which the disc orbits, which will usually be dominated by the central object.

Assuming that Φ is axisymmetric, the ϕ-component of this equation, multiplied by r, is

$$\frac{\partial}{\partial t}(\rho r u_\phi) + \frac{1}{r}\frac{\partial}{\partial r}[r^2(\rho u_r u_\phi - T_{r\phi})] + \frac{1}{r}\frac{\partial}{\partial \phi}[r(\rho u_\phi^2 - T_{\phi\phi})] + \frac{\partial}{\partial z}[r(\rho u_z u_\phi - T_{z\phi})] = 0.$$

Integrate this equation over the cylinder $\mathscr{C}_r$ of radius r, again assuming no loss or gain through the vertical boundaries, and assuming (to be examined later) that $ru_\phi = h(r)$ from orbital dynamics:

$$\frac{\partial}{\partial t}(\mathscr{M}h) + \frac{\partial}{\partial r}(\mathscr{F}h + \mathscr{G}) = 0,$$

where

$$\mathscr{G}(r,t) = -\int_{\mathscr{C}_r} rT_{r\phi}\,dA.$$

The radial flux of angular momentum, $\mathscr{F}h + \mathscr{G}$, is the sum of two parts:

- advection of orbital advection ($\mathscr{F}h$) by the accretion flow
- an internal torque ($\mathscr{G}$) due to collective effects

4.2. Diffusion equation for mass evolution

Since h depends only on r, our two 1D conservation equations are

$$\frac{\partial \mathscr{M}}{\partial t} + \frac{\partial \mathscr{F}}{\partial r} = 0, \qquad \frac{\partial \mathscr{M}}{\partial t}h + \frac{\partial}{\partial r}(\mathscr{F}h + \mathscr{G}) = 0.$$

Eliminate $\mathscr{M}$ to obtain

$$\mathscr{F}\frac{dh}{dr} + \frac{\partial \mathscr{G}}{\partial r} = 0,$$

which determines $\mathscr{F}$ instantaneously. Therefore $\mathscr{M}$ evolves according to

$$\frac{\partial \mathscr{M}}{\partial t} = \frac{\partial}{\partial r}\left[\left(\frac{dh}{dr}\right)^{-1}\frac{\partial \mathscr{G}}{\partial r}\right].$$

The physical interpretation of this analysis is as follows. Since the motion is assumed to be dominated by circular orbital motion in the midplane, the specific angular momentum of a fluid element determines its orbital radius through the function $h(r)$ (increasing for stable orbits). Any radial transport of angular momentum ($\mathscr{G}$) implies a radial transport of mass ($\mathscr{F}$). Therefore the evolution of the mass distribution of the disc is governed by the transport of angular momentum.

A more usual notation refers instead to the *surface density* $\Sigma(r,t)$, the *mean radial velocity* $\bar{u}_r(r,t)$ and the *mean effective kinematic viscosity* $\bar{\nu}(r,t)$, related via

$$\mathscr{M} = 2\pi r\Sigma, \qquad \mathscr{F} = 2\pi r\Sigma\bar{u}_r, \qquad \mathscr{G} = -2\pi\bar{\nu}\Sigma r^3\frac{d\Omega}{dr}.$$

These can be defined by

$$\Sigma = \int_{-\infty}^{\infty}\langle\rho\rangle\,dz, \qquad \Sigma\bar{u}_r = \int_{-\infty}^{\infty}\langle\rho u_r\rangle\,dz, \qquad \bar{\nu}\Sigma r\frac{d\Omega}{dr} = \int_{-\infty}^{\infty}\langle T_{r\phi}\rangle\,dz,$$

where

$$\langle\cdot\rangle = \frac{1}{2\pi}\int_0^{2\pi}\cdot\,d\phi$$

is an azimuthal average (if required). Here the internal torque is being represented as if it were a viscous torque resulting from the orbital shear.

In these variables, we obtain

$$\frac{\partial\Sigma}{\partial t} = \frac{1}{r}\frac{\partial}{\partial r}\left[\left(\frac{dh}{dr}\right)^{-1}\frac{\partial}{\partial r}\left(-\bar{\nu}\Sigma r^3\frac{d\Omega}{dr}\right)\right].$$

For a Keplerian disc, with $\Omega \propto r^{-3/2}$ and $h = r^2\Omega \propto r^{1/2}$, this equation simplifies to

$$\frac{\partial\Sigma}{\partial t} = \frac{3}{r}\frac{\partial}{\partial r}\left[r^{1/2}\frac{\partial}{\partial r}\left(r^{1/2}\bar{\nu}\Sigma\right)\right],$$

which is a *diffusion equation for the surface density.*

Exercise: Using specific angular momentum $h(r)$ as a spatial variable instead of r, show that

$$\mathscr{F} = -\frac{\partial\mathscr{G}}{\partial h}, \qquad \frac{\partial\mathscr{M}}{\partial t} = \frac{dh}{dr}\frac{\partial^2\mathscr{G}}{\partial h^2}.$$

If $\bar{\nu}$ depends on r only, show that we obtain a diffusion equation in the form

$$\frac{\partial\mathscr{G}}{\partial t} = \left(-\bar{\nu}r^2\frac{d\Omega}{dr}\frac{dh}{dr}\right)\frac{\partial^2\mathscr{G}}{\partial h^2}.$$

A narrow ring spreads diffusively because viscous or frictional processes transport angular momentum outwards from the more rapidly rotating inner part to the less rapidly rotating outer part. The inner part loses angular momentum and spreads inwards, while the outer part gains angular momentum and spreads outwards.

[FIGURE]

4.3. Evolution of orbital energy

Recall that $d\varepsilon = \Omega\, dh$ for circular orbits.

Consider

$$\begin{aligned}\frac{\partial}{\partial t}(\mathscr{M}\varepsilon) + \frac{\partial}{\partial r}(\mathscr{F}\varepsilon) &= \varepsilon\left(\frac{\partial\mathscr{M}}{\partial t} + \frac{\partial\mathscr{F}}{\partial r}\right) + \mathscr{F}\frac{d\varepsilon}{dr} \\ &= 0 + \mathscr{F}\Omega\frac{dh}{dr} \\ &= -\Omega\frac{\partial\mathscr{G}}{\partial r}.\end{aligned}$$

Therefore

$$\frac{\partial}{\partial t}(\mathscr{M}\varepsilon) + \frac{\partial}{\partial r}(\mathscr{F}\varepsilon + \mathscr{G}\Omega) = \mathscr{G}\frac{d\Omega}{dr}.$$

Here $\mathscr{G}\Omega$ is a radial energy flux associated with the internal torque. The RHS of this equation is minus the rate of dissipation of orbital energy per unit radius.

In the more usual notation (dividing through by $2\pi r$ to get quantities per unit area), this equation becomes

$$\frac{\partial}{\partial t}(\Sigma\varepsilon) + \frac{1}{r}\frac{\partial}{\partial r}\left(r\Sigma\bar{u}_r\varepsilon - \bar{\nu}\Sigma r^3\frac{d\Omega}{dr}\Omega\right) = -\bar{\nu}\Sigma r^2\left(\frac{d\Omega}{dr}\right)^2.$$

Assuming that the dissipated energy is converted into heat and lost locally by blackbody radiation, the surface temperature $T_{\mathrm{s}}(r,t)$ of the disc is given by

$$2\sigma T_{\mathrm{s}}^4 = \bar{\nu}\Sigma r^2\left(\frac{d\Omega}{dr}\right)^2.$$

More generally this defines the *effective temperature* $T_{\mathrm{eff}}(r,t)$.

4.4. Viscosity

Possible contributors to the stress $T_{r\phi}$ include:

- magnetic fields: $\dfrac{B_r B_\phi}{\mu_0}$ (for sufficiently ionized discs)
- self-gravity: $-\dfrac{g_r g_\phi}{4\pi}$ (for sufficiently massive discs)
- fluctuating velocities due to waves, instabilities or turbulence: $-\langle\rho u_r' u_\phi'\rangle$
- true viscous stress: $-\rho\nu r\dfrac{d\Omega}{dr}$ (rarely significant)

Consideration of angular-momentum transport processes and the local vertical structure of the disc (see later) leads plausibly to a relation of the form

$$\bar{\nu} = \bar{\nu}(r,\Sigma) \qquad \text{(e.g. double power law).}$$

- If $\bar{\nu} = \bar{\nu}(r)$ only, the diffusion equation is *linear*
- If $\bar{\nu} = \bar{\nu}(r,\Sigma)$, the diffusion equation is *nonlinear*

Exercise: Suppose that non-zero fluxes of mass and angular momentum through the vertical boundaries are permitted. Show that, if $S(r,t)$ and $T(r,t)$ are the rates at which mass and angular momentum are added to the disc per unit area (azimuthally averaged, if necessary), then the diffusion equation is modified to

$$\frac{\partial\Sigma}{\partial t} = \frac{1}{r}\frac{\partial}{\partial r}\left\{\left(\frac{dh}{dr}\right)^{-1}\left[\frac{\partial}{\partial r}\left(-\bar{\nu}\Sigma r^3\frac{d\Omega}{dr}\right) + r(Sh - T)\right]\right\} + S.$$

Lecture 5: Boundary conditions and steady accretion

5.1. Boundary conditions

At a *free boundary*, which can move radially as the disc spreads, the internal torque vanishes:

$$\mathscr{G} = 0.$$

A spreading disc will reach $r = 0$ in a finite time, whereas the outer edge might expand indefinitely. The *inner boundary condition* is then particularly important, and depends on the nature of the central object.

Black hole

Circular orbits are unstable sufficiently close to the event horizon (see Example 1.3).

e.g. for a non-rotating (Schwarzschild) black hole:

$$\Omega^2 = \frac{GM}{r^3}, \qquad \Omega_r^2 = \Omega^2 \left(1 - \frac{6\,GM}{c^2 r}\right),$$

so orbits are unstable for

$$r < r_{\mathrm{ms}} = \frac{6\,GM}{c^2} = 3\,r_{\mathrm{h}}.$$

This critical radius is referred to as the *marginally stable circular orbit* or *innermost stable circular orbit* (ISCO). Actually this orbit is unstable. Given a radial displacement, the particle rapidly spirals into the black hole without needing to lose any angular momentum.

In a fluid disc, a rapid transition occurs near r_{ms}. The radial velocity $|\bar{u}_r|$ increases very rapidly as r decreases near r_{ms}, so Σ decreases very rapidly to conserve the radial mass flux.

It is expected that the torque $\mathscr{G} \approx 0$ at $r = r_{\mathrm{in}} \approx r_{\mathrm{ms}}$.

Exercise: Using Newtonian dynamics in the model potential $\Phi = -GM/(R - r_{\mathrm{h}})$, calculate ε and h for a circular orbit of radius $r = r_{\mathrm{ms}} = 3r_{\mathrm{h}}$. Use these conserved quantities to work out how the radial and azimuthal velocities depend on r for a particle that is infinitesimally displaced inwards from this circular orbit.

Star with a negligible magnetic field

In this case the disc may extend to the stellar surface $r = R_*$. In contrast to the disc, the star (supported mainly by pressure) usually rotates at only a small fraction of the Keplerian rate:

$$\Omega_* \ll \sqrt{\frac{GM}{R_*^3}}.$$

Ω makes a rapid adjustment from the Keplerian value to the stellar value in a *boundary layer*:

The usual argument is that the 'viscous' torque $\mathscr{G} = 0$ at $r = r_{\text{in}} \approx R_*$ where $\dfrac{d\Omega}{dr} = 0$.

Star with a significant magnetic field

In this case the disc is disrupted within the *magnetospheric radius*, leading to the formation of a cavity and accretion along field lines towards the magnetic poles of the star. This *polar accretion* is observed in many young stars and neutron stars.

The star may exert a torque on the disc if they are linked by magnetic field lines. The outcome of this complicated process depends on the stellar rotation rate and magnetic field, the accretion rate, etc.

Treatment for $r \gg r_{\text{in}}$

If we are mainly interested in scales $r \gg r_{\text{in}}$, we may formally let $r_{\text{in}} \to 0$.

To allow a mass flux at the origin but no torque:

$$\mathscr{F} \to \text{constant}, \qquad \mathscr{G} \to 0 \qquad \text{as } r \to 0.$$

To allow a torque at the origin:

$$\mathscr{G} \to \text{constant} \qquad \text{as } r \to 0.$$

For a Keplerian disc:

$$\mathscr{F} \propto r\Sigma\bar{u}_r \propto r^{1/2}\frac{\partial}{\partial r}\left(r^{1/2}\bar{\nu}\Sigma\right), \qquad \mathscr{G} \propto r^{1/2}\bar{\nu}\Sigma.$$

In the first case (no torque at the origin): $\bar{\nu}\Sigma \to \text{constant}$ as $r \to 0$.

In the second case (torque at the origin): $r^{1/2}\bar{\nu}\Sigma \to \text{constant}$ as $r \to 0$.

5.2. Steady accretion discs

In a steady state, mass conservation gives

$$\mathscr{F} = -\dot{M} = \text{constant},$$

where $\dot{M}$ is the *mass accretion rate*. (We neglect the slow variation of the potential due to the increase in M.) Angular momentum conservation gives

$$\mathscr{F}h + \mathscr{G} = \text{constant}.$$

If the inner boundary condition is $\mathscr{G} = 0$ at $r = r_{\text{in}}$, the solution is

$$\mathscr{G} = \dot{M}(h - h_{\text{in}}), \qquad h_{\text{in}} = h(r_{\text{in}}).$$

For a Keplerian disc, we have $\mathscr{G} = 3\pi\bar{\nu}\Sigma h$ and $h \propto r^{1/2}$, so

$$\bar{\nu}\Sigma = \frac{\dot{M}}{3\pi}\left(1 - \sqrt{\frac{r_{\text{in}}}{r}}\right).$$

If we know the function $\bar{\nu}(r, \Sigma)$, this provides a complete solution for the disc.

From the expression for the surface temperature,

$$2\sigma T_{\text{s}}^4 = \bar{\nu}\Sigma r^2\left(\frac{d\Omega}{dr}\right)^2,$$

in the case of a steady Keplerian disc we have

$$\sigma T_{\text{s}}^4 = \frac{3GM\dot{M}}{8\pi r^3}\left(1 - \sqrt{\frac{r_{\text{in}}}{r}}\right).$$

The total luminosity of a disc extending to $r_{\rm out} = \infty$ is

$$L_{\rm disc} = \int_{r_{\rm in}}^{\infty} \frac{3GM\dot{M}}{4\pi r^3} \left(1 - \sqrt{\frac{r_{\rm in}}{r}}\right) 2\pi r \, dr = \frac{1}{2} \frac{GM\dot{M}}{r_{\rm in}}.$$

This is exactly equal to the rate at which orbital binding energy is transferred to the gas as it passes from to $r_{\rm in}$. But it is only half the potential energy released. In the case of accretion on to a stellar surface, the remaining energy is released in the boundary layer.

The spectrum of a disc emitting blackbody radiation is a stretched version of the blackbody spectrum, with the high-energy radiation coming mainly from the inner disc and the low-energy radiation coming mainly from the outer disc.

Exercise: Show that, at radius r in a steady Keplerian accretion disc, the ratio of the rate of energy dissipation to the rate at which orbital energy is given up by the inflowing matter is

$$3\left(1 - \sqrt{\frac{r_{\rm in}}{r}}\right).$$

(The fact that this ratio is not equal to unity everywhere is explained by the fact that the internal torque provides an outward flux of energy as well as angular momentum.)

Lecture 6: Time-dependent accretion

We return to the diffusion equation governing the spreading of a Keplerian disc,

$$\frac{\partial\Sigma}{\partial t} = \frac{3}{r}\frac{\partial}{\partial r}\left[r^{1/2}\frac{\partial}{\partial r}\left(r^{1/2}\bar{\nu}\Sigma\right)\right].$$

Time-dependent solutions illustrate the mechanics of accretion.

6.1. Linear diffusion equation

The linear case, in which $\bar{\nu} = \bar{\nu}(r)$, can be treated using *Green's function.*

Let $\Delta(r, r_0, t)$ be the solution $\mathscr{M}(r,t) = 2\pi r\Sigma(r,t)$ of the diffusion equation with the initial condition $\mathscr{M}(r,0) = \delta(r - r_0)$, i.e. a very narrow ring of radius r_0 and unit mass. Then the solution for any initial condition $\mathscr{M}(r,0) = \mathscr{M}_0(r)$ and time $t > 0$ is

$$\mathscr{M}(r,t) = \int_0^\infty \Delta(r, r_0, t)\mathscr{M}_0(r_0)\, dr_0,$$

by linear superposition.

It is possible to calculate $\Delta(r, r_0, t)$ in terms of Bessel functions for any power law $\bar{\nu} \propto r^a$ and any boundary conditions. These functions become elementary if $a = (1+4n)/(1+2n)$ for some integer n.

The easiest special case for illustration is $\bar{\nu} \propto r$. Let

$$\bar{\nu} = Ar, \qquad y = \sqrt{\frac{4r}{3A}} \quad (\propto h), \qquad g = r^{1/2}\bar{\nu}\Sigma = Ar^{3/2}\Sigma = \frac{Ar^{1/2}\mathscr{M}}{2\pi} \quad (\propto \mathscr{G}),$$

to obtain the classical diffusion equation (see Example 1.4),

$$\frac{\partial g}{\partial t} = \frac{\partial^2 g}{\partial y^2}.$$

The fundamental solution of the diffusion equation, corresponding to the initial condition $g(y,0) = \delta(y)$ and no boundary conditions, is

$$g(y,t) = \frac{1}{\sqrt{4\pi t}}\exp\left(-\frac{y^2}{4t}\right).$$

The solution representing a spreading ring with zero torque ($g = 0$) at the inner boundary $r = r_{\mathrm{in}}$ (corresponding to $y = y_{\mathrm{in}}$) and with initial radius $r_0 > r_{\mathrm{in}}$ (corresponding to $y = y_0$) is therefore

$$g(y,t) = \frac{C}{\sqrt{4\pi t}}\left\{\exp\left[-\frac{(y-y_0)^2}{4t}\right] - \exp\left[-\frac{(y+y_0-2y_{\mathrm{in}})^2}{4t}\right]\right\}.$$

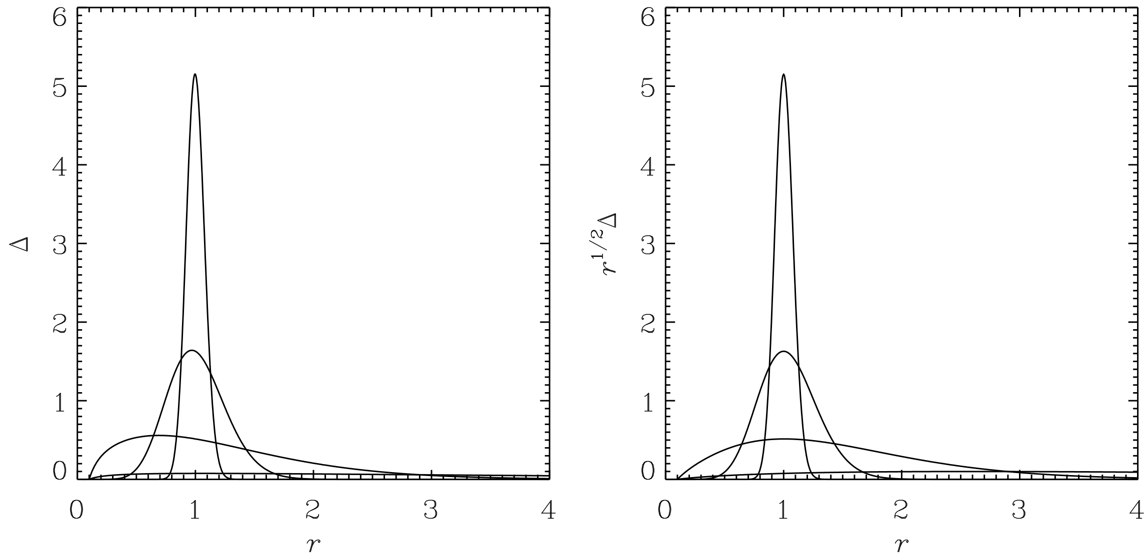
This uses a superposition of translations of the fundamental solution to construct a solution satisfying the boundary condition by the method of images.

The initial condition

$$\mathscr{M} = \delta(r - r_0) \qquad \Rightarrow \qquad g = \frac{Ar_0^{1/2}}{2\pi}\delta(y - y_0)\frac{1}{\sqrt{3Ar_0}}$$

(the last factor coming from dy/dr at $r = r_0$) gives $2\pi C = \sqrt{A/3}$, so Green's function is

$$\Delta(r, r_0, t) = \frac{1}{\sqrt{12\pi Art}} \left\{ \exp\left[-\frac{(\sqrt{r} - \sqrt{r_0})^2}{3At} \right] - \exp\left[-\frac{(\sqrt{r} + \sqrt{r_0} - 2\sqrt{r_{\rm in}})^2}{3At} \right] \right\}.$$



Radial distributions of mass (left) and angular momentum (right) at times $t = 0.001, 0.01, 0.1, 1$ for a spreading ring with viscosity $\bar{\nu} = Ar$, in units such that $A = 1$ and $r_0 = 1$. The inner boundary condition is that the torque vanishes at $r_{\rm in} = 0.1$.

Angular momentum is transported outwards and taken up by a diminishing fraction of the initial mass moving to larger and larger radii.

In the limit of large time, $t \gg r_0/A$, we obtain

$$\begin{aligned}
\Delta &\approx \frac{1}{\sqrt{12\pi Art}} \left\{ \left[1 - \frac{(\sqrt{r} - \sqrt{r_0})^2}{3At} \right] - \left[1 - \frac{(\sqrt{r} + \sqrt{r_0} - 2\sqrt{r_{\rm in}})^2}{3At} \right] \right\} \\
2\pi r \Sigma &\approx \frac{1}{\sqrt{12\pi Art}} \frac{1}{3At} \left(2\sqrt{r} - 2\sqrt{r_{\rm in}} \right) \left(2\sqrt{r_0} - 2\sqrt{r_{\rm in}} \right) \\
\bar{\nu}\Sigma &\approx \frac{1}{3\pi} \frac{\left(\sqrt{r_0} - \sqrt{r_{\rm in}} \right)}{\sqrt{3\pi A t^3}} \left(1 - \sqrt{\frac{r_{\rm in}}{r}} \right),
\end{aligned}$$

which looks like the solution for a steady disc, but with declining accretion rate $\left(\sqrt{r_0} - \sqrt{r_{\rm in}}\right)/\sqrt{3\pi A t^3}$.

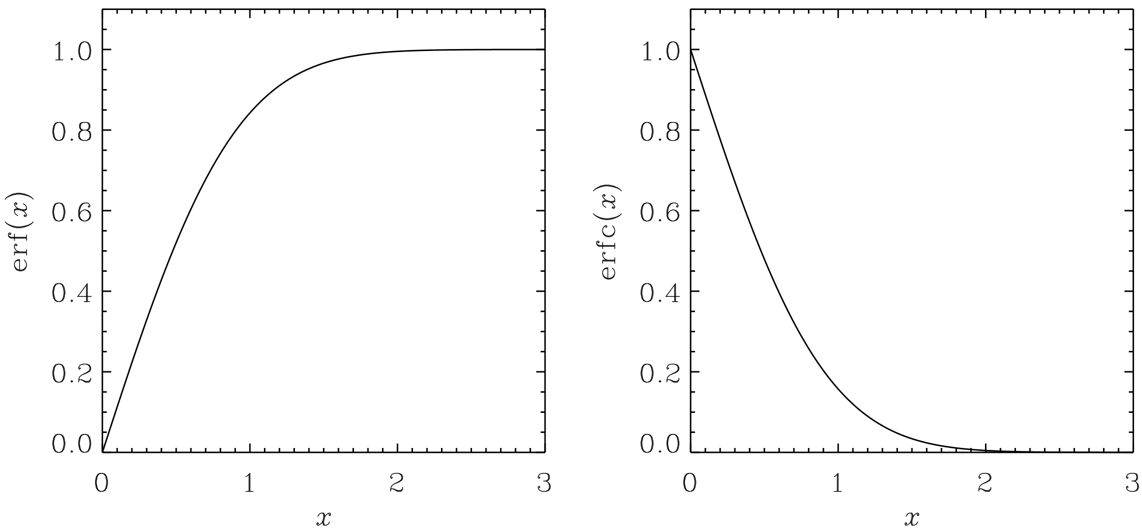
Exercise: Show that, in the joint limit of large radius, $r \gg r_0, r_{\rm in}$, and large time, $t \gg r_0/A$, we obtain a similarity solution (see later) of the form

$$\Delta \approx \frac{2(\sqrt{r_0} - \sqrt{r_{\rm in}})\, e^{-r/3At}}{\sqrt{\pi (3At)^3}}.$$

The *error function* $\mathrm{erf}(x)$ and the *complementary error function* $\mathrm{erfc}(x)$ are defined by

$$\mathrm{erf}(x) = \frac{1}{\sqrt{\pi}} \int_{-x}^{x} e^{-\xi^2}\, d\xi = \frac{2}{\sqrt{\pi}} \int_0^x e^{-\xi^2}\, d\xi, \qquad \mathrm{erfc}(x) = 1 - \mathrm{erf}(x) = \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-\xi^2}\, d\xi.$$

We have $\mathrm{erf}(x) \sim (2/\sqrt{\pi})x$ for $|x| \ll 1$ and $\mathrm{erfc}(x) \sim (1/\sqrt{\pi})x^{-1}\, e^{-x^2}$ for $x \gg 1$.



The mass remaining in the disc at time t is (substitute $\sqrt{r} = \sqrt{r_0} + \sqrt{3At}\,\xi$ in the first term and $\sqrt{r} = 2\sqrt{r_{\text{in}}} - \sqrt{r_0} + \sqrt{3At}\,\xi$ in the second term, giving $dr = 2\sqrt{r}\,\sqrt{3At}\,d\xi$)

$$\int_{r_{\text{in}}}^{\infty} \Delta(r, r_0, t)\,dr = \frac{1}{\sqrt{\pi}}\int_{-\xi_{\text{in}}}^{\infty} e^{-\xi^2}\,d\xi - \frac{1}{\sqrt{\pi}}\int_{\xi_{\text{in}}}^{\infty} e^{-\xi^2}\,d\xi = \frac{1}{\sqrt{\pi}}\int_{-\xi_{\text{in}}}^{\xi_{\text{in}}} e^{-\xi^2}\,d\xi = \operatorname{erf}(\xi_{\text{in}}),$$

where $\xi_{\text{in}} = (\sqrt{r_0} - \sqrt{r_{\text{in}}})/\sqrt{3At}$. This declines $\propto t^{-1/2}$ for large t.

The radial mass flux is

$$\mathscr{F} = -6\pi r^{1/2}\frac{\partial}{\partial r}\left(r^{1/2}\bar{\nu}\Sigma\right) = -6\pi\sqrt{\frac{1}{3A}}\frac{\partial g}{\partial y},$$

giving an accretion rate at r_{in}:

$$\dot{M}_{\text{in}}(t) = -\mathscr{F}(r_{\text{in}}, t) = \frac{1}{\sqrt{\pi}}\frac{\xi_{\text{in}}}{t}\exp(-\xi_{\text{in}}^2).$$

This does indeed integrate to give (note that $\xi_{\text{in}} \propto t^{-1/2}$, so $dt/t = -2\,d\xi_{\text{in}}/\xi_{\text{in}}$)

$$\int_0^t \dot{M}_{\text{in}}(t')\,dt' = \operatorname{erfc}(\xi_{\text{in}}).$$

The angular momentum remaining in the disc is

$$\begin{aligned}
&\int_{r_{\text{in}}}^{\infty} \sqrt{GMr}\,\Delta(r, r_0, t)\,dr \\
&= \frac{\sqrt{GM}}{\sqrt{\pi}}\int_{-\xi_{\text{in}}}^{\infty}\left(\sqrt{r_0} + \sqrt{3At}\,\xi\right)e^{-\xi^2}\,d\xi - \frac{1}{\sqrt{\pi}}\int_{\xi_{\text{in}}}^{\infty}\left(2\sqrt{r_{\text{in}}} - \sqrt{r_0} + \sqrt{3At}\,\xi\right)e^{-\xi^2}\,d\xi \\
&= \frac{\sqrt{GM}}{\sqrt{\pi}}\int_{-\infty}^{\xi_{\text{in}}}\left(\sqrt{r_0} - \sqrt{3At}\,\xi\right)e^{-\xi^2}\,d\xi - \frac{1}{\sqrt{\pi}}\int_{\xi_{\text{in}}}^{\infty}\left(2\sqrt{r_{\text{in}}} - \sqrt{r_0} + \sqrt{3At}\,\xi\right)e^{-\xi^2}\,d\xi \\
&= h_0 - h_{\text{in}}\operatorname{erfc}(\xi_{\text{in}}),
\end{aligned}$$

i.e. the initial angular momentum minus that accreted through r_{in}.

6.2. Nonlinear diffusion equation

Consider a power-law viscosity

$$\bar{\nu} = A r^a \Sigma^b, \qquad A = \text{constant},$$

and let $r_{\text{in}} \to 0$.

The problem is then scale-free and admits special algebraic *similarity solutions* (see Examples 1.5 and 1.6). These are generally attractors for solutions of the initial-value problem. Unlike the linear case, these solutions may have free boundaries beyond which the density vanishes. If the torque vanishes at the origin, then the total angular momentum is conserved, while the total mass of the disc declines as a power-law in time.

The conserved angular momentum is

$$\sqrt{GM}\, C, \qquad C = \int \sqrt{r}\, \Sigma\, 2\pi r\, dr.$$

Dimensional analysis gives

$$[A] = M^{-b} L^{2-a+2b} T^{-1}, \qquad [C] = M L^{1/2}.$$

Using C, A and t (the time elapsed since the formation of the disc), we can construct a time-dependent characteristic length-scale $R(t)$ given by

$$R^{2-a+(5/2)b} = C^b A t.$$

Thus

$$R \propto t^{2/(4-2a+5b)}, \qquad M_{\text{disc}} \propto \frac{C}{R^{1/2}} \propto t^{-1/(4-2a+5b)}.$$

A spreading ring with a power-law viscosity tends towards the similarity solution long after the ring has spread to the inner boundary and forgotten its initial radius.

Exercise: Verify that the linear diffusion equation with $\bar{\nu} = A r^a$ $(a < 2)$ admits similarity solutions

$$\Sigma \propto R^{-5/2} \xi^{-a} \exp\left[-\frac{\xi^{2-a}}{(2-a)^2}\right]$$

(with conserved angular momentum) and

$$\Sigma \propto R^{-2} \xi^{-(1/2)-a} \exp\left[-\frac{\xi^{2-a}}{(2-a)^2}\right]$$

(with conserved mass), where the similarity variable is $\xi = r/R(t)$ with $R^{2-a} = 3At$.

Lecture 7: Vertical structure

7.1. Hydrostatic equilibrium

The dominant force balance in the z direction perpendicular to the plane of the disc is

$$0 = -\rho\frac{\partial\Phi}{\partial z} - \frac{\partial p}{\partial z}.$$

The Taylor expansion of Φ about the midplane is $\Phi(r,z) = \Phi(r,0) + \frac{1}{2}\Phi_{zz}(r,0)z^2 + \cdots$, so the vertical gravity in a thin disc is

$$g_z = -\frac{\partial\Phi}{\partial z} \approx -\Phi_{zz}(r,0)z = -\Omega_z^2 z,$$

where $\Omega_z(r)$ is the vertical frequency (recall that $\Omega_z = \Omega$ for a Keplerian disc).

The *equation of vertical hydrostatic equilibrium* is therefore

$$\frac{\partial p}{\partial z} = -\rho\Omega_z^2 z.$$

This is essentially an ordinary differential equation in z at each r (and ϕ and t). [In the local approximation the ODE $\frac{dp}{dz} = -\rho\Omega_z^2 z$ is exact for hydrostatic solutions independent of (x,y,t).]

As in a star, pressure supports the disc against gravity in the vertical direction. But note that the disc is centrifugally supported in the radial direction.

This analysis is for a *non-self-gravitating* disc, using the vertical gravity due to the central object. In a *self-gravitating disc* the disc makes an additional contribution to g_z and affects the hydrostatic structure.

If p and ρ are related in a known way, we can solve for the hydrostatic structure. e.g. for an isothermal ideal gas, $p = c_\mathrm{s}^2\rho$, where c_s = constant is the isothermal sound speed. Then the solution is a Gaussian:

$$p \propto \rho \propto \exp\left(-\frac{z^2}{2H^2}\right),$$

with scaleheight

$$H = \frac{c_\mathrm{s}}{\Omega_z} \qquad \left(= \frac{c_\mathrm{s}}{\Omega} \quad \text{for Keplerian}\right).$$

Formally, the disc extends to $z = \pm\infty$, and the thin-disc approximation breaks down once z/r is no longer small, but there is essentially no mass at such heights.

7.2. Hydrostatic models

More generally, define the *surface density* Σ, *vertically integrated pressure* P and *scaleheight* H by

$$\Sigma = \int \rho\, dz, \qquad P = \int p\, dz, \qquad \Sigma H^2 = \int \rho z^2\, dz,$$

where the integrals are over the full vertical extent of disc. H can be interpreted as the standard deviation of the mass distribution. Note that (assuming boundary conditions $zp \to 0$ as $z \to \pm\infty$)

$$P = \int 1 \cdot p\, dz = [zp] - \int z \frac{dp}{dz}\, dz = 0 + \int z\rho\Omega_z^2 z\, dz = \Sigma H^2 \Omega_z^2.$$

We can reduce the problem to a dimensionless form:

$$\rho(z) = \hat{\rho} \cdot \tilde{\rho}(\tilde{z}), \qquad p(z) = \hat{p} \cdot \tilde{p}(\tilde{z}),$$

where

$$\hat{\rho} = \frac{\Sigma}{H}, \qquad \hat{p} = \frac{P}{H}$$

are characteristic values of density and pressure, while $\tilde{\rho}$ and $\tilde{p}$ are dimensionless functions of the dimensionless coordinate

$$\tilde{z} = \frac{z}{H}.$$

These satisfy the dimensionless equation of hydrostatic equilibrium,

$$\frac{d\tilde{p}}{d\tilde{z}} = -\tilde{\rho}\tilde{z},$$

and the normalization conditions

$$\int \tilde{\rho}\, d\tilde{z} = \int \tilde{p}\, d\tilde{z} = \int \tilde{\rho}\tilde{z}^2\, d\tilde{z} = 1.$$

The *isothermal model* ($p \propto \rho$) is the solution

$$\tilde{\rho} = \tilde{p} = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{\tilde{z}^2}{2}\right).$$

The *uniform model* ($\rho =$ constant) is the solution

$$\tilde{\rho} = \begin{cases} \frac{1}{2\sqrt{3}}, & \tilde{z}^2 < 3, \\ 0, & \tilde{z}^2 > 3, \end{cases}, \qquad \tilde{p} = \begin{cases} \frac{1}{4\sqrt{3}}(3 - \tilde{z}^2), & \tilde{z}^2 < 3, \\ 0, & \tilde{z}^2 > 3. \end{cases}$$

The *polytropic model* of index n ($p \propto \rho^{1+1/n}$, where $n > 0$ is not necessarily an integer) is the solution

$$\tilde{\rho} = C_\rho \left(1 - \frac{\tilde{z}^2}{2n+3}\right)^n, \qquad \tilde{p} = C_p \left(1 - \frac{\tilde{z}^2}{2n+3}\right)^{n+1},$$

(valid for $\tilde{z}^2 < 2n + 3$ only, otherwise $\tilde{\rho} = \tilde{p} = 0$), with normalizing constants

$$C_\rho = \frac{\Gamma(n + \frac{3}{2})}{\Gamma(n+1)} \frac{1}{\sqrt{(2n+3)\pi}}, \qquad C_p = \frac{(n + \frac{3}{2})}{(n+1)} C_\rho = \frac{\Gamma(n + \frac{5}{2})}{\Gamma(n+2)} \frac{1}{\sqrt{(2n+3)\pi}}.$$

Here

$$\Gamma(p) = \int_0^\infty x^{p-1}\, e^{-x}\, dx, \qquad p > 0$$

is the Gamma function, equal to $(p-1)!$ for integers p. A useful integral here is

$$\int_{-1}^{1} (1-x^2)^p\, dx = \frac{\sqrt{\pi}\,\Gamma(p+1)}{\Gamma(p+\frac{3}{2})}, \qquad p > -1.$$

It can be shown that the polytropic model reduces to the uniform model in the limit $n \to 0$ and reduces to the isothermal model in the limit $n \to \infty$.

While the isothermal model extends formally to $z = \pm\infty$, the other models have definite surfaces beyond which there is a vacuum.

7.3. Order-of-magnitude estimates and time-scales

Here we consider simple scaling relations ($\sim$), omitting numerical factors of order unity.

An important dimensionless parameter of a thin disc is the *aspect ratio*

$$\frac{H}{r} \ll 1.$$

From hydrostatic equilibrium,

$$\frac{\partial p}{\partial z} = -\rho\Omega_z^2 z \qquad \Rightarrow \qquad \frac{p}{H} \sim \rho\Omega^2 H \qquad \Rightarrow \qquad c_\mathrm{s} \sim \Omega H,$$

where $c_\mathrm{s} = \sqrt{p/\rho}$ is the isothermal sound speed.

The dimensions of dynamic viscosity

$$[\rho\nu] = ML^{-1}T^{-1}$$

are the same of those of p/Ω. We write

$$\rho\nu = \frac{\alpha p}{\Omega},$$

where α is the *dimensionless viscosity parameter.* If α is regarded as a constant, this relation is known as the *alpha viscosity prescription.* Then

$$\nu = \frac{\alpha c_\mathrm{s}^2}{\Omega} \sim \alpha c_\mathrm{s} H.$$

In the kinetic theory of gases, the kinematic viscosity is $\nu \sim v\ell$, where v is the mean speed of the molecules and ℓ is their mean free path. This molecular viscosity is negligible for astrophysical discs. But a similar estimate can be made for the effective 'eddy viscosity' of turbulence, if v is a typical turbulent velocity and ℓ is the correlation length of the turbulence. For subsonic turbulence with $v \lesssim c_\mathrm{s}$ and $\ell \lesssim H$, we expect that $\alpha \lesssim 1$.

The stress is then

$$T_{r\phi} = \rho\nu r\frac{d\Omega}{dr} = -q\alpha p.$$

The idea behind $|T_{r\phi}| \sim \alpha p$ is that, whatever physical process gives rise to the stress, it should scale with the pressure. This assumption is probably correct for local processes such as small-scale turbulent motions resulting from instabilities (see later).

Three important characteristic time-scales in a disc can be defined:

Dynamical time-scale (time-scale of orbital motion and of vertical hydrostatic equilibrium):

$$t_{\text{dyn}} \sim \frac{1}{\Omega} \sim \frac{H}{c_{\text{s}}}.$$

Viscous time-scale (time-scale of radial motion and of evolution of the surface density):

$$t_{\text{visc}} \sim \frac{r^2}{\bar{\nu}} \sim \alpha^{-1}\left(\frac{H}{r}\right)^{-2} t_{\text{dyn}}.$$

Thermal time-scale (time-scale of vertical thermal balance):

$$t_{\text{th}} \sim \frac{\text{internal energy/area}}{\text{dissipation rate/area}} \sim \frac{P}{\bar{\nu}\Sigma\Omega^2} \sim \frac{c_{\text{s}}^2}{\bar{\nu}\Omega^2} \sim \frac{H^2}{\bar{\nu}} \sim \alpha^{-1} t_{\text{dyn}}.$$

For a thin disc with $\alpha < 1$, we have the hierarchy

$$t_{\text{dyn}} < t_{\text{th}} \ll t_{\text{visc}}.$$

Furthermore, all three time-scales usually increase with r.

The Mach number of the orbital motion is

$$\text{Ma} \sim \frac{r\Omega}{c_{\text{s}}} \sim \left(\frac{H}{r}\right)^{-1}.$$

The typical accretion velocity is

$$|\bar{u}_r| \sim \frac{\bar{\nu}}{r} \sim \alpha\left(\frac{H}{r}\right) c_{\text{s}}.$$

For a thin disc, we have the hierarchy

$$|\bar{u}_r| \ll c_{\text{s}} \ll r\Omega,$$

so the orbital motion is highly supersonic while the accretion flow is highly subsonic.

The relative contribution of the radial pressure gradient to the radial component of the equation of motion is

$$\frac{\partial p}{\partial r}\Big/\rho r\Omega^2 \sim \frac{\rho c_{\text{s}}^2}{r}\Big/\rho r\Omega^2 \sim \frac{c_{\text{s}}^2}{r^2\Omega^2} \sim \left(\frac{H}{r}\right)^2.$$

Vertical variations of the radial gravitational acceleration are also of this order. Other terms in the radial equation of motion, such as inertial terms associated with the radial motion, are smaller still. Therefore

$$u_\phi = r\Omega\left[1 + O\left(\frac{H}{r}\right)^2\right],$$

so treating the azimuthal fluid velocity as equal to the orbital velocity of a test particle is an excellent approximation for a thin disc. In general, the thin-disc approximations involve fractional errors of $O(H/r)^2$. A formal asymptotic treatment of thin discs is possible, using as small parameter a characteristic value of $(H/r)^2$.

Exercise: If $\Phi = -GM/R$ and $p/\rho = c_{\text{s}}^2 = \epsilon^2 GM/r$ ('locally isothermal'), where $R = \sqrt{r^2 + z^2}$ and $\epsilon = \text{constant}$, show that *exact* force balances are achieved in all directions if

$$\rho = f(r)\exp\left(\frac{r - R}{\epsilon^2 R}\right), \qquad \Omega^2 = \frac{GM}{r^3}\left[\frac{r}{R} - \epsilon^2\left(1 - \frac{d\ln f}{d\ln r}\right)\right].$$

Lecture 8: Radiative models

8.1. Equations of vertical structure

A *radiative model* describes the vertical structure of a disc in which the energy dissipated by viscosity is carried away by radiation from the surfaces of the disc.

The energy flux due to radiative diffusion, for an optically thick disc, is

$$\mathbf{F} = -\frac{16\sigma T^3}{3\kappa\rho}\boldsymbol{\nabla}T,$$

where κ is the (Rosseland mean) *opacity*. The dominant balance in the thermal energy equation, for a thin disc, is

$$0 = \rho\nu\left(r\frac{d\Omega}{dr}\right)^2 - \frac{\partial F_z}{\partial z}.$$

Contributions from F_r and from radial advection are smaller by $O(H/r)^2$, e.g.

$$\left(\frac{1}{\gamma-1}\right)\bar{u}_r\frac{\partial p}{\partial r}\bigg/\rho\nu\left(r\frac{d\Omega}{dr}\right)^2 \sim \frac{\bar{u}_r\rho c_\mathrm{s}^2}{r}\bigg/\rho\bar{\nu}\Omega^2 \sim \frac{c_\mathrm{s}^2}{r^2\Omega^2} \sim \frac{H^2}{r^2}.$$

The equations of vertical structure for a radiative, Keplerian disc are then

$$\begin{aligned}
\frac{\partial p}{\partial z} &= -\rho\Omega^2 z,\\
\frac{\partial F_z}{\partial z} &= \frac{9}{4}\rho\nu\Omega^2,\\
F_z &= -\frac{16\sigma T^3}{3\kappa\rho}\frac{\partial T}{\partial z},
\end{aligned}$$

together with an *equation of state*, e.g.

$$p = \frac{\mathscr{R}\rho T}{\mu} + \frac{4\sigma T^4}{3c} \qquad \text{(ideal gas + radiation)}$$

(where $\mathscr{R}$ is the gas constant and μ the mean molecular weight), an opacity function $\kappa(\rho, T)$, a viscosity prescription for $\bar{\nu}$ and boundary conditions, e.g. the 'zero boundary conditions' $\rho = p = T = 0$ at $z = \pm z_\mathrm{s}$ (or, more realistically, matching to an atmospheric model at the photosphere).

The problem is analogous to the radial structure of a star. In the local approximation, these are exact ODEs for equilibrium solutions independent of (x, y, t).

Opacity is often approximated by a power law, e.g. *Thomson opacity* (due to electron scattering: hotter regions of ionized discs)

$$\kappa = \text{constant} \approx 0.33\,\mathrm{cm}^2\,\mathrm{g}^{-1},$$

or *Kramers opacity* (due to free-free/bound-free transitions: cooler regions of ionized discs)

$$\kappa = C_\kappa\rho T^{-7/2}, \qquad C_\kappa \approx 4.5\times10^{24}\,\mathrm{cm}^5\,\mathrm{g}^{-2}\,\mathrm{K}^{7/2}.$$

In cooler discs, dust and molecules dominate the opacity.

8.2. Radiative, Keplerian disc with gas pressure, power-law opacity and alpha viscosity

We aim to solve

$$\frac{dp}{dz} = -\rho\Omega^2 z, \qquad \frac{dF_z}{dz} = \frac{9}{4}\rho\nu\Omega^2, \qquad F_z = -\frac{16\sigma T^3}{3\kappa\rho}\frac{dT}{dz},$$

with

$$p = \frac{\mathscr{R}\rho T}{\mu}, \qquad \kappa = C_\kappa \rho^x T^y, \qquad \rho\nu = \frac{\alpha p}{\Omega}.$$

The problem can be reduced to a dimensionless form by writing

$$\rho(z) = \hat{\rho}\cdot\tilde{\rho}(\tilde{z}), \qquad p(z) = \hat{p}\cdot\tilde{p}(\tilde{z}), \qquad T(z) = \hat{T}\cdot\tilde{T}(\tilde{z}), \qquad F_z(z) = \hat{F}\cdot\tilde{F}(\tilde{z}),$$

with dimensionless vertical coordinate $\tilde{z} = z/H$ and characteristic values

$$\hat{\rho} = \frac{\Sigma}{H}, \qquad \hat{p} = \frac{P}{H}, \qquad \hat{T} = \frac{\mu}{\mathscr{R}}\frac{\hat{p}}{\hat{\rho}}, \qquad \hat{F} = \frac{16\sigma\hat{T}^4}{3\hat{\tau}},$$

where, again, $P = \Sigma H^2\Omega^2$, and the characteristic optical thickness is

$$\hat{\tau} = \hat{\kappa}\Sigma = C_\kappa \hat{\rho}^x \hat{T}^y \Sigma.$$

We obtain the dimensionless equations of vertical structure,

$$\frac{d\tilde{p}}{d\tilde{z}} = -\tilde{\rho}\tilde{z}, \qquad \frac{d\tilde{F}}{d\tilde{z}} = \lambda\tilde{p}, \qquad \frac{d\tilde{T}}{d\tilde{z}} = -\tilde{\rho}^{x+1}\tilde{T}^{y-3}\tilde{F}, \qquad \tilde{p} = \tilde{\rho}\tilde{T},$$

subject to the normalization conditions

$$\int \tilde{\rho}\, d\tilde{z} = \int \tilde{p}\, d\tilde{z} = \int \tilde{\rho}\tilde{z}^2\, d\tilde{z} = 1$$

and the zero boundary conditions $\tilde{\rho} = \tilde{p} = \tilde{T} = 0$ at the surfaces $\tilde{z} = \pm\tilde{z}_{\mathrm{s}}$.

The inclusion of thermal physics leads to a specific solution of the problem of hydrostatic equilibrium considered in the previous lecture.

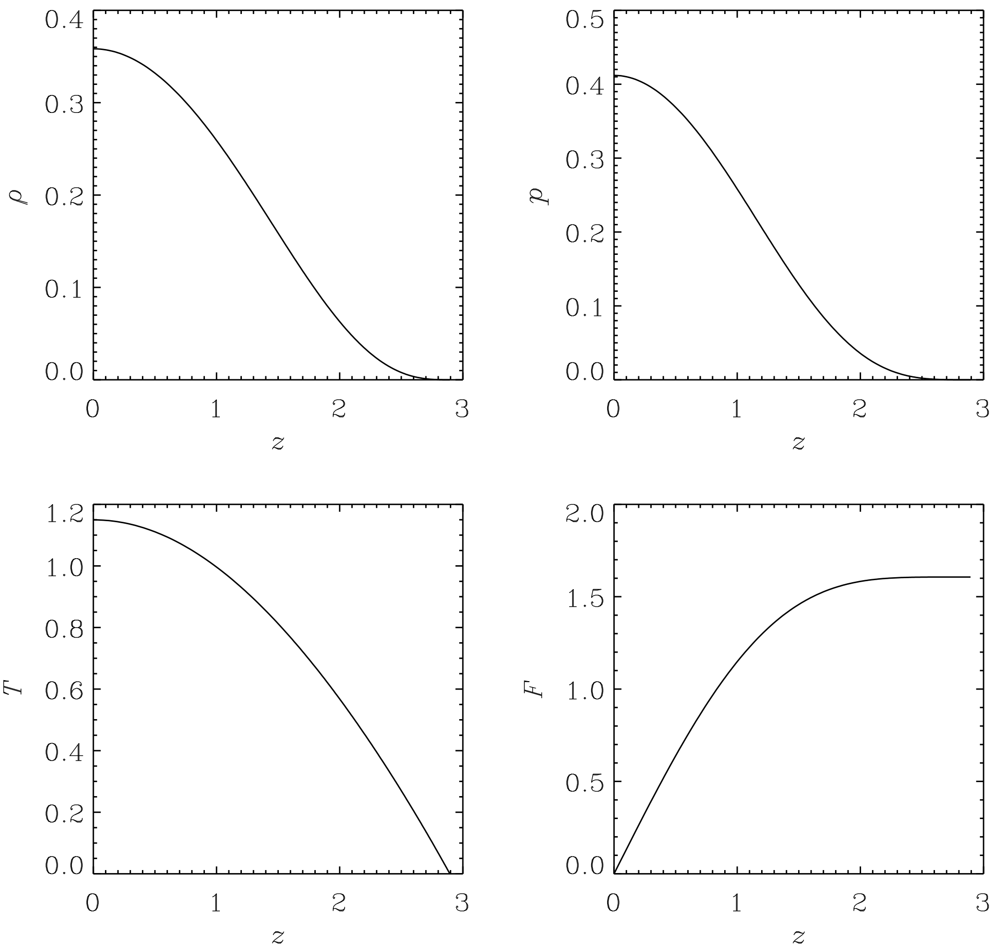
λ is an eigenvalue of the problem, which can be interpreted as the dimensionless cooling rate:

$$2\tilde{F}_{\mathrm{s}} = \int \frac{d\tilde{F}}{d\tilde{z}}\, d\tilde{z} = \lambda \int \tilde{p}\, d\tilde{z} = \lambda.$$

The numerical solution for Thomson opacity ($x = y = 0$) gives $\lambda = 3.213$ and $z_{\mathrm{s}} = 2.895$. This solution is similar to a polytropic model with $n \approx 2.4$.

The energy equation in a steady state gives the equilibrium condition for thermal balance:

$$\begin{aligned}
\frac{\hat{F}}{H}\lambda &= \frac{9}{4}\alpha\Omega\hat{p} \\
\lambda &= \frac{9}{4}\alpha\Omega\frac{P}{\hat{F}} \\
&= \frac{27}{64}\alpha\frac{\Sigma H^2\Omega^3\hat{\tau}}{\sigma\hat{T}^4} \\
&= \frac{27}{64}\alpha\frac{C_\kappa}{\sigma}\left(\frac{\mathscr{R}}{\mu}\right)^{4-y} H^{-x+2y-6}\Omega^{2y-5}\Sigma^{x+2}.
\end{aligned}$$



The vertically integrated viscosity is

$$\bar{\nu}\Sigma = \int \rho\nu\,dz = \int \frac{\alpha p}{\Omega}\,dz = \frac{\alpha P}{\Omega} = \alpha\Sigma H^2\Omega.$$

Thermal balance implies

$$H^{x-2y+6} \propto \Omega^{2y-5}\Sigma^{x+2},$$

so

$$\begin{aligned}\bar{\nu} &\propto H^2\Omega \\ &\propto \Omega^{(x+2y-4))/(x-2y+6)}\Sigma^{2(x+2)/(x-2y+6)} \\ &\propto r^{-3(x+2y-4)/2(x-2y+6)}\Sigma^{2(x+2)/(x-2y+6)}.\end{aligned}$$

e.g. for Thomson opacity ($x = y = 0$),

$$\bar{\nu} \propto r\Sigma^{2/3},$$

or for Kramers opacity ($x = 1$, $y = -7/2$),

$$\bar{\nu} \propto r^{15/14}\Sigma^{3/7}.$$

The constant of proportionality involves a numerical coefficient and various powers of C_κ/σ, $\mathscr{R}/\mu$ and GM.

The heating and cooling rates per unit area (equal in a thermal steady state) are

$$\mathscr{H} = \int \frac{9}{4}\alpha\Omega p\,dz = \frac{9}{4}\alpha\Omega P,$$

$$\mathscr{C} = 2\sigma T_{\mathrm{s}}^4 = 2F_z\Big|_{z=z_{\mathrm{s}}} = 2\hat{F}\tilde{F}_{\mathrm{s}} = \lambda\frac{16\sigma\hat{T}^4}{3\hat{\tau}}.$$

Note that

$$T_{\mathrm{s}}^4 = \frac{8\lambda}{3\hat{\tau}}\hat{T}^4,$$

so $\hat{T} \gg T_{\mathrm{s}}$ in a highly optically thick disc (as assumed when using zero boundary conditions).

8.3. Viscous instability

Consider the nonlinear diffusion equation for a Keplerian disc,

$$\frac{\partial\Sigma}{\partial t} = \frac{3}{r}\frac{\partial}{\partial r}\left[r^{1/2}\frac{\partial}{\partial r}\left(r^{1/2}\bar{\nu}\Sigma\right)\right],$$

with $\bar{\nu} = \bar{\nu}(r,\Sigma)$. Linearize about any given solution $\Sigma_0(r,t)$:

$$\Sigma(r,t) = \Sigma_0(r,t) + \Sigma'(r,t), \qquad |\Sigma'| \ll \Sigma_0,$$

so that

$$(\bar{\nu}\Sigma)' = \frac{\partial(\bar{\nu}\Sigma)}{\partial\Sigma}\Sigma' = \beta\bar{\nu}\Sigma', \qquad \beta = \frac{\partial\ln(\bar{\nu}\Sigma)}{\partial\ln\Sigma}.$$

We then obtain the linearized diffusion equation,

$$\frac{\partial\Sigma'}{\partial t} = \frac{3}{r}\frac{\partial}{\partial r}\left[r^{1/2}\frac{\partial}{\partial r}\left(r^{1/2}\beta\bar{\nu}\Sigma'\right)\right].$$

The evolution is unstable (antidiffusive) for $\beta < 0$: perturbations grow rapidly on short length-scales, causing the disc to break into rings. Astrophysical applications may be complicated by thermal instability, which often coincides with it and dominates (see next lecture).

Exercise: Show that, when $\beta < 0$, short-wavelength perturbations Σ' with radial wavenumber k grow in time at the rate $3|\beta|\bar{\nu}k^2$, when $1/k$ is small (as in the WKB approximation) compared to the characteristic scale on which the solution Σ_0 varies. (The diffusion equation itself becomes inaccurate when $1/k$ is not large compared to H, so the fastest-growing modes typically have wavelengths of a few times H.)

Lecture 9: Thermal instability / Hydrodynamics of the shearing sheet

9.1. Thermal instability

So far, we have assumed a balance between heating and cooling: $\frac{9}{4}\bar{\nu}\Sigma\Omega^2 = \mathscr{H} = \mathscr{C} = 2F_{\rm s}$.

Now relax this assumption, but assume that $\alpha \ll 1$ so that $t_{\rm dyn} \ll t_{\rm th} \ll t_{\rm visc}$. Consider behaviour on the timescale $t_{\rm th}$; we can then assume that the disc is hydrostatic and that the surface density does not evolve.

By solving the equations of vertical structure *except* thermal balance, we can calculate $\mathscr{H}$ and $\mathscr{C}$ as functions of $(\Sigma, \bar{\nu}\Sigma)$. In fact $\mathscr{H}$ depends only on $\bar{\nu}\Sigma$. The equation of thermal balance $\mathscr{H} = \mathscr{C}$ defines a curve in the $(\Sigma, \bar{\nu}\Sigma)$ plane.

Along the equilibrium curve, $d\mathscr{H} = d\mathscr{C}$ and $d(\bar{\nu}\Sigma) = \beta\bar{\nu}\,d\Sigma$, where $\beta = \left(\frac{\partial\ln(\bar{\nu}\Sigma)}{\partial\ln\Sigma}\right)_r$:

$$\frac{d\mathscr{H}}{d(\bar{\nu}\Sigma)}\,d(\bar{\nu}\Sigma) = \frac{\partial\mathscr{C}}{\partial\Sigma}\,d\Sigma + \frac{\partial\mathscr{C}}{\partial(\bar{\nu}\Sigma)}\,d(\bar{\nu}\Sigma)$$
$$\frac{d\mathscr{H}}{d(\bar{\nu}\Sigma)} = \frac{1}{\beta\bar{\nu}}\frac{\partial\mathscr{C}}{\partial\Sigma} + \frac{\partial\mathscr{C}}{\partial(\bar{\nu}\Sigma)}.$$

The internal energy content of disc per unit area is $\sim P \sim (\Omega/\alpha)\bar{\nu}\Sigma$. If some heat is added, $\bar{\nu}\Sigma$ increases but Σ is fixed on the timescale $t_{\rm th}$. The system is thermally unstable if the excess heating outweighs the excess cooling, i.e. if

$$\frac{d\mathscr{H}}{d(\bar{\nu}\Sigma)} > \frac{\partial\mathscr{C}}{\partial(\bar{\nu}\Sigma)}, \qquad \text{i.e. if} \qquad \frac{1}{\beta\bar{\nu}}\frac{\partial\mathscr{C}}{\partial\Sigma} > 0.$$

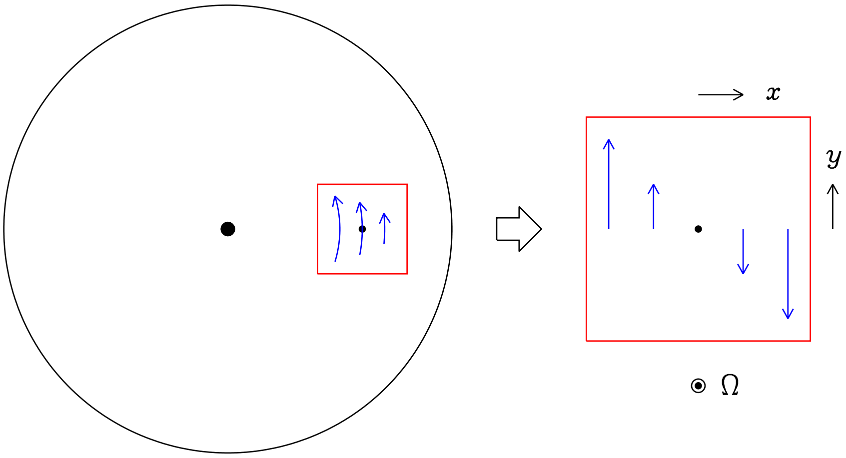
In practice $\partial\mathscr{C}/\partial\Sigma < 0$ (because, at fixed $\bar{\nu}\Sigma$, $\Sigma \propto 1/\bar{\nu} \propto 1/(\alpha T)$, and $\mathscr{C}$ generally increases with T), so thermal instability occurs (like viscous instability) when $\beta < 0$. Thermal instability then dominates (as its timescale is shorter).

9.2. Outbursts

We have seen that a radiative disc with gas pressure and Thomson opacity has $\bar{\nu}\Sigma \propto r\Sigma^{5/3}$ and is viscously and thermally stable. For cooler discs undergoing H ionization, the graph of $\bar{\nu}\Sigma$ versus Σ can involve an ‘S curve’, leading to instability and limit-cycle behaviour, which explains the outbursts in many cataclysmic variables, X-ray binaries and other systems.

9.3. Hydrodynamics of the shearing sheet

Recall the local view of an astrophysical disc: a linear shear flow $\mathbf{u}_0 = -Sx\,\mathbf{e}_y$ in a frame rotating with $\boldsymbol{\Omega} = \Omega\,\mathbf{e}_z$. Here Ω and $S = -r\,d\Omega/dr$ are evaluated at the reference radius r_0.



The model is either horizontally unbounded or equipped with (modified) periodic boundary conditions (see later). Possible treatments of the vertical structure are:

- ignore z completely (2D shearing sheet)
- neglect vertical gravity: homogeneous in z
- include vertical gravity: isothermal, uniform, polytropic, radiative, etc. models

9.4. Homogeneous incompressible fluid

Consider a 3D model, unbounded or periodic in (x, y, z), with a uniform kinematic viscosity ν.

The equation of motion is

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}\cdot\boldsymbol{\nabla}\mathbf{u} + 2\boldsymbol{\Omega}\times\mathbf{u} = -\boldsymbol{\nabla}\Phi_{\mathrm{t}} - \frac{1}{\rho}\boldsymbol{\nabla}p + \nu\nabla^2\mathbf{u},$$

subject to the incompressibility condition

$$\boldsymbol{\nabla}\cdot\mathbf{u} = 0.$$

The basic state is $\mathbf{u} = \mathbf{u}_0 = -Sx\,\mathbf{e}_y$, with hydrostatic pressure $p = p_0(z)$. There is a uniform viscous stress, but it has no divergence and causes no accretion flow.

Introduce perturbations (not necessarily small):

$$\mathbf{u} = \mathbf{u}_0 + \mathbf{v}(\mathbf{x}, t), \qquad p = p_0 + \rho\psi(\mathbf{x}, t).$$

Then

$$\frac{\partial \mathbf{v}}{\partial t} + \mathbf{u}_0\cdot\boldsymbol{\nabla}\mathbf{v} + \mathbf{v}\cdot\boldsymbol{\nabla}\mathbf{u}_0 + \mathbf{v}\cdot\boldsymbol{\nabla}\mathbf{v} + 2\boldsymbol{\Omega}\times\mathbf{v} = -\boldsymbol{\nabla}\psi + \nu\nabla^2\mathbf{v},$$

$$\boldsymbol{\nabla}\cdot\mathbf{v} = 0.$$

In components:

$$\left(\frac{\partial}{\partial t} - Sx\frac{\partial}{\partial y} + \mathbf{v}\cdot\boldsymbol{\nabla}\right) v_x - 2\Omega v_y = -\frac{\partial\psi}{\partial x} + \nu\nabla^2 v_x,$$

$$\left(\frac{\partial}{\partial t} - Sx\frac{\partial}{\partial y} + \mathbf{v}\cdot\boldsymbol{\nabla}\right) v_y + (2\Omega - S)v_x = -\frac{\partial\psi}{\partial y} + \nu\nabla^2 v_y,$$

$$\left(\frac{\partial}{\partial t} - Sx\frac{\partial}{\partial y} + \mathbf{v}\cdot\boldsymbol{\nabla}\right) v_z = -\frac{\partial\psi}{\partial z} + \nu\nabla^2 v_z.$$

Consider a plane-wave solution in the form of a *shearing wave* :

$$\mathbf{v}(\mathbf{x},t) = \mathrm{Re}\left\{\tilde{\mathbf{v}}(t)\,\exp[i\mathbf{k}(t)\cdot\mathbf{x}]\right\},$$

$$\psi(\mathbf{x},t) = \mathrm{Re}\left\{\tilde{\psi}(t)\,\exp[i\mathbf{k}(t)\cdot\mathbf{x}]\right\},$$

with time-dependent wavevector $\mathbf{k}(t)$. Then

$$\left(\frac{\partial}{\partial t} - Sx\frac{\partial}{\partial y}\right)\mathbf{v} = \mathrm{Re}\left\{\left[\frac{d\tilde{\mathbf{v}}}{dt} + \left(i\frac{d\mathbf{k}}{dt}\cdot\mathbf{x} - Sx\,ik_y\right)\tilde{\mathbf{v}}\right]\exp[i\mathbf{k}(t)\cdot\mathbf{x}]\right\}.$$

If we choose

$$\frac{d\mathbf{k}}{dt} = Sk_y\,\mathbf{e}_x,$$

then two terms cancel and we are left with $\dfrac{d\tilde{\mathbf{v}}}{dt}$.

This means

$$k_x = k_{x0} + Sk_y t, \qquad k_y = \text{constant}, \qquad k_z = \text{constant}.$$

Tilting of the wavefronts by the shear flow, and

Dual shear flow in Fourier space:

Furthermore, the nonlinear term vanishes:

$$\begin{aligned}
\mathbf{v}\cdot\boldsymbol{\nabla}\mathbf{v} &= \mathrm{Re}\left[\tilde{\mathbf{v}}\,e^{i\mathbf{k}\cdot\mathbf{x}}\right]\cdot\boldsymbol{\nabla}\mathrm{Re}\left[\tilde{\mathbf{v}}\,e^{i\mathbf{k}\cdot\mathbf{x}}\right]\\
&= \mathrm{Re}\left[\mathbf{k}\cdot\tilde{\mathbf{v}}\,e^{i\mathbf{k}\cdot\mathbf{x}}\right]\mathrm{Re}\left[i\tilde{\mathbf{v}}\,e^{i\mathbf{k}\cdot\mathbf{x}}\right]\\
&= 0,
\end{aligned}$$

because $\boldsymbol{\nabla}\cdot\mathbf{v}=0$ implies $i\mathbf{k}\cdot\tilde{\mathbf{v}}=0$. (This is a special result for an incompressible fluid. Note also that the nonlinear term does not vanish for a superposition of shearing waves.)

The amplitude equations for a shearing wave are

$$\begin{aligned}
\frac{d\tilde{v}_x}{dt}-2\Omega\tilde{v}_y &= -ik_x\tilde{\psi}-\nu k^2\tilde{v}_x,\\
\frac{d\tilde{v}_y}{dt}+(2\Omega-S)\tilde{v}_x &= -ik_y\tilde{\psi}-\nu k^2\tilde{v}_y,\\
\frac{d\tilde{v}_z}{dt} &= -ik_z\tilde{\psi}-\nu k^2\tilde{v}_z,\\
i\mathbf{k}\cdot\tilde{\mathbf{v}} &= 0,
\end{aligned}$$

with $k^2=|\mathbf{k}|^2$.

The viscous terms can be taken care of by a viscous decay factor

$$\begin{aligned}
E_\nu(t) &= \exp\left(-\int\nu k^2\,dt\right)\\
&= \exp\left\{-\nu\left[(k_{x0}^2+k_y^2+k_z^2)t+Sk_{x0}k_yt^2+\frac{1}{3}S^2k_y^2t^3\right]\right\}.
\end{aligned}$$

The decay is faster than exponential if $k_y\neq 0$.

Write $\tilde{\mathbf{v}}=E_\nu(t)\hat{\mathbf{v}}(t)$ and $\tilde{\psi}=E_\nu(t)\hat{\psi}(t)$ to eliminate the ν terms in the amplitude equations. Then eliminate variables in favour of $\hat{v}_x$ to obtain (see Example 2.1)

$$\frac{d^2}{dt^2}\left(k^2\hat{v}_x\right)+\Omega_r^2k_z^2\hat{v}_x=0,$$

where $\Omega_r^2=2\Omega(2\Omega-S)$ is the square of the epicyclic frequency in the local approximation.

Summary of outcomes (see Example 2.1):

- Stable if $\Omega_r^2>0$: $|\hat{\mathbf{v}}|^2$ oscillates if $k_y=0$, or decays algebraically if $k_y\neq 0$.
- Unstable if $\Omega_r^2<0$: $|\hat{\mathbf{v}}|^2$ grows exponentially if $k_y=0$, or grows algebraically if $k_y\neq 0$.

When $\nu>0$, E_ν kills off any algebraic growth for $k_y\neq 0$. But axisymmetric disturbances ($k_y=0$) of sufficiently large scale grow exponentially.

We conclude that a rotating shear flow is linearly stable when $\Omega_r^2>0$, but unstable when $\Omega_r^2<0$.

This agrees with the stability of circular test-particle orbits. It also agrees with *Rayleigh's criterion* for the linear stability of a cylindrical shear flow $\mathbf{u}=r\Omega(r)\,\mathbf{e}_\phi$ to axisymmetric perturbations: the flow is unstable if the specific angular momentum $|r^2\Omega|$ decreases outwards.

The case $\Omega_r^2=0$ (either a non-rotating shear flow or one with uniform specific angular momentum) is marginally Rayleigh-stable and allows algebraic growth in the absence of viscosity.

Lecture 10: Vortices in discs

10.1. The vorticity equation

To study the behaviour of vortices in discs we consider a 2D incompressible sheet. Velocity perturbations in the plane of the disc satisfy the nonlinear equations

$$\left(\frac{\partial}{\partial t} - Sx\frac{\partial}{\partial y} + \mathbf{v}\cdot\boldsymbol{\nabla}\right)v_x - 2\Omega v_y = -\frac{\partial\psi}{\partial x} + \nu\nabla^2 v_x,$$

$$\left(\frac{\partial}{\partial t} - Sx\frac{\partial}{\partial y} + \mathbf{v}\cdot\boldsymbol{\nabla}\right)v_y + (2\Omega - S)v_x = -\frac{\partial\psi}{\partial y} + \nu\nabla^2 v_y,$$

$$\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} = 0.$$

Introduce the *streamfunction* $\chi(x, y, t)$ such that

$$v_x = \frac{\partial\chi}{\partial y}, \qquad v_y = -\frac{\partial\chi}{\partial x}.$$

The instantaneous streamlines are the curves $\chi = \text{constant}$. The vorticity perturbation is

$$\boldsymbol{\nabla}\times\mathbf{v} = \left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right)\mathbf{e}_z = (-\nabla^2\chi)\,\mathbf{e}_z = \zeta\,\mathbf{e}_z.$$

Take the curl of the equation of motion to eliminate ψ: many terms cancel, leaving the *vorticity equation*

$$\left(\frac{\partial}{\partial t} - Sx\frac{\partial}{\partial y} + \mathbf{v}\cdot\boldsymbol{\nabla}\right)\zeta = \nu\nabla^2\zeta, \qquad (1)$$

a nonlinear advection–diffusion equation to be solved in conjunction with Poisson's equation

$$\nabla^2\chi = -\zeta.$$

The total *absolute vorticity* is $(2\Omega - S + \zeta)\,\mathbf{e}_z$, with contributions from background rotation, background shear and the velocity perturbation.

Note that the Coriolis force drops out of the 2D incompressible dynamics, so the fact that the sheet is rotating is irrelevant. This model is too constrained to allow epicyclic/inertial oscillations; it involves pure vortex dynamics with background shear.

Multiply equation (1) by ζ to obtain an equation for the *enstrophy* $\frac{1}{2}\zeta^2$:

$$\begin{aligned}\left(\frac{\partial}{\partial t} - Sx\frac{\partial}{\partial y} + \mathbf{v}\cdot\boldsymbol{\nabla}\right)\left(\frac{1}{2}\zeta^2\right) &= \nu\zeta\nabla^2\zeta \\ &= \boldsymbol{\nabla}\cdot(\nu\zeta\boldsymbol{\nabla}\zeta) - \nu|\boldsymbol{\nabla}\zeta|^2.\end{aligned}$$

Integrated over an area A, assuming suitable boundary conditions, this equation implies that the enstrophy decays:

$$\frac{d}{dt}\int\frac{1}{2}\zeta^2\,dA = -\int\nu|\boldsymbol{\nabla}\zeta|^2\,dA.$$

To maintain vorticity perturbations in the presence of viscosity requires baroclinic or 3D effects, or other source terms.

10.2. Zonal flows

Axisymmetric structures in the vorticity correspond to y-independent solutions of equation (1). These have $v_x = 0$ and satisfy the diffusion equation

$$\frac{\partial \zeta}{\partial t} = \nu \frac{\partial^2 \zeta}{\partial x^2}.$$

They involve a purely *zonal flow* $v_y(x,t)$ ('zonal' = 'azimuthal') and are unaffected by background shear. To the extent that viscosity is negligible, they are equilibrium solutions. They involve a 'geostrophic' balance between the Coriolis force and a radial pressure gradient.

10.3. Shearing vortices

Shearing-wave solutions of equation (1),

$$\zeta(\mathbf{x},t) = \mathrm{Re}\left\{\tilde{\zeta}(t)\, \exp[i\mathbf{k}(t)\cdot\mathbf{x}]\right\},$$

satisfy the amplitude equation

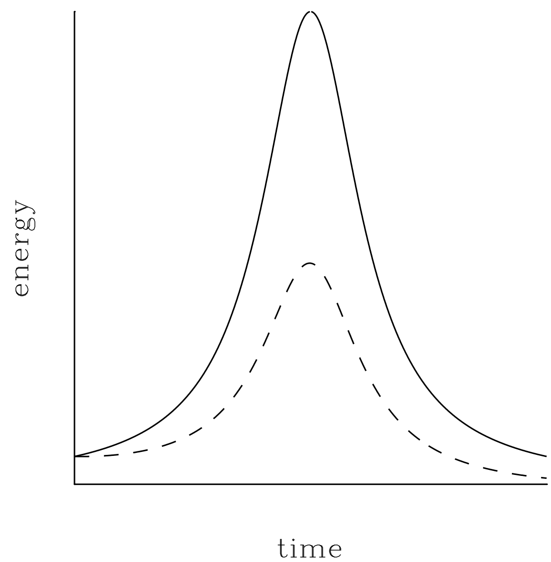
$$\frac{d\tilde{\zeta}}{dt} = -\nu k^2 \tilde{\zeta}.$$

The nonlinear term $\mathbf{v}\cdot\boldsymbol{\nabla}\zeta$ vanishes because $\boldsymbol{\nabla}\cdot\mathbf{v} = 0$ implies $i\mathbf{k}\cdot\tilde{\mathbf{v}} = 0$. So the vorticity amplitude decays viscously:

$$\tilde{\zeta} \propto E_\nu(t).$$

The kinetic energy can undergo *transient growth*:

$$|\tilde{\mathbf{v}}|^2 \propto k^2|\tilde{\chi}|^2 \propto k^2 \left|\frac{\tilde{\zeta}}{k^2}\right|^2 \propto \frac{E_\nu^2}{k^2}.$$



10.4. Elliptical vortex patches

In the absence of viscosity, the vorticity equation (1) reduces to

$$\frac{D\zeta}{Dt} = 0.$$

Consider a uniform vortex patch defined by a closed contour, inside which $\zeta = \zeta_0$, a non-zero constant, and outside which $\zeta = 0$.

The vorticity perturbation ζ generates a velocity field $\mathbf{v}$ that, together with the background shear $-Sx\,\mathbf{e}_y$, advects the contour. Do steady solutions exist in which the flow induced by the vortex resists the shear?

Consider an elliptical vortex patch (centred on the origin WLOG), with semi-axes a and b, inclined at an angle θ with respect to y and x axes).

As shown in Example 2.3, the velocity $\mathbf{v}$ induced by ζ_0 causes the ellipse to rotate with angular velocity

$$\dot{\theta} = \frac{ab\zeta_0}{(a+b)^2},$$

while the background shear $-Sx\,\mathbf{e}_y$ deforms the ellipse according to

$$\frac{\dot{a}}{a} = -\frac{\dot{b}}{b} = S\sin\theta\cos\theta, \qquad \dot{\theta} = \frac{S(b^2\cos^2\theta - a^2\sin^2\theta)}{a^2 - b^2}.$$

Combine these effects to obtain

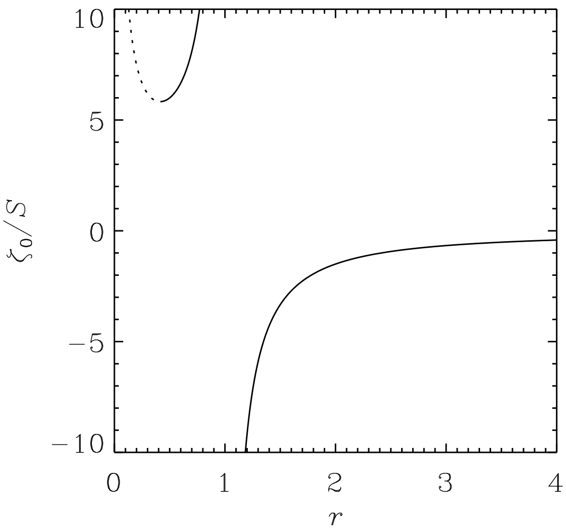
$$\begin{aligned}
\frac{\dot{a}}{a} &= -\frac{\dot{b}}{b} = S\sin\theta\cos\theta, \\
\dot{\theta} &= \frac{S(b^2\cos^2\theta - a^2\sin^2\theta)}{a^2 - b^2} + \frac{ab\,\zeta_0}{(a+b)^2}.
\end{aligned}$$

Note that the area πab is conserved, as expected. Rewrite in terms of the aspect ratio $r = a/b$:

$$\begin{aligned}
\frac{\dot{r}}{r} &= 2S\sin\theta\cos\theta, \\
\dot{\theta} &= \frac{S(\cos^2\theta - r^2\sin^2\theta)}{r^2 - 1} + \frac{r\,\zeta_0}{(r+1)^2}.
\end{aligned}$$

Equilibrium solutions representing steady vortices have $\theta = 0$ WLOG (let $r < 1$ if necessary) and

$$\frac{S}{r^2 - 1} + \frac{r\,\zeta_0}{(r+1)^2} = 0 \qquad \Rightarrow \qquad \frac{\zeta_0}{S} = -\frac{(r+1)}{r(r-1)} = f(r).$$



In the context of a rotating disc (and assuming $S/\Omega > 0$), vortices are called *cyclonic* if $\zeta_0/S > 0$ (vorticity in the same sense as rotation) and *anticyclonic* if $\zeta_0/S < 0$.

The linearized equations governing the stability of an equilibrium vortex ($\theta = 0$) are

$$\dot{\delta r} = 2Sr\,\delta\theta, \qquad \dot{\delta\theta} = S\frac{\partial g}{\partial r}\,\delta r,$$

where

$$g\left(r, \frac{\zeta_0}{S}\right) = \frac{1}{r^2 - 1} + \frac{r}{(r+1)^2}\frac{\zeta_0}{S}$$

vanishes at equilibrium, where its derivative is (**exercise**)

$$\frac{\partial g}{\partial r} = \frac{2 - (r+1)^2}{r(r^2-1)^2} = -\frac{r}{(r+1)^2}\frac{df}{dr}.$$

So

$$\ddot{\delta r} = 2S^2 r\frac{\partial g}{\partial r}\,\delta r,$$

which implies instability for

$$\frac{\partial g}{\partial r} > 0, \qquad \text{i.e.} \qquad \frac{df}{dr} < 0, \qquad \text{i.e.} \qquad r < \sqrt{2} - 1.$$

Vortices have elliptical streamlines and are susceptible to the *elliptical instability* in 3D. In a Keplerian disc, sufficiently strong anticyclonic vortices with $r < 4$ are vigorously unstable (through violation of a Rayleigh-like criterion). Weaker anticyclonic vortices with $r > 4$ can exist in a Keplerian disc; these tend to have weaker forms of elliptical instability involving resonant destabilization of inertial waves, which may produce turbulent motion without destroying the vortex.

Exercise: The velocity field inside the vortex has a linear dependence on the Cartesian coordinates. Show that it has the form (including the contribution from background shear)

$$\mathbf{u} = \frac{S}{r-1}\left(\frac{y}{r}, -rx\right).$$

Lecture 11: Density waves and gravitational instability

11.1. The compressible shearing sheet

Now consider, in the local model, a 2D compressible sheet that is self-gravitating but inviscid. (See Example 2.4 for the effects of viscosity.) The sheet has velocity $\mathbf{u}(x,y,t)$, surface density $\Sigma(x,y,t)=\int\rho\,dz$ and 2D pressure $P(x,y,t)=\int p\,dz$, satisfying the equation of mass conservation,

$$\frac{\partial\Sigma}{\partial t}+\boldsymbol{\nabla}\cdot(\Sigma\mathbf{u})=0,$$

and the equation of motion,

$$\frac{\partial\mathbf{u}}{\partial t}+\mathbf{u}\cdot\boldsymbol{\nabla}\mathbf{u}+2\boldsymbol{\Omega}\times\mathbf{u}=-\boldsymbol{\nabla}\Phi_{\mathrm{t,m}}-\boldsymbol{\nabla}\Phi_{\mathrm{d,m}}-\frac{1}{\Sigma}\boldsymbol{\nabla}P,$$

where the tidal potential in the midplane is $\Phi_{\mathrm{t,m}}=-\Omega Sx^2$, the disc potential $\Phi_{\mathrm{d}}(x,y,z,t)$ satisfies Poisson's equation

$$\nabla^2\Phi_{\mathrm{d}}=4\pi G\Sigma\,\delta(z),$$

and its value in the midplane is $\Phi_{\mathrm{d,m}}(x,y,t)=\Phi_{\mathrm{d}}(x,y,0,t)$.

To avoid the complications of thermal physics and focus on the dynamics, we assume a barotropic relation $P=P(\Sigma)$.

(These equations are only a model; they cannot be derived exactly from the true 3D equations.)

Poisson's equation can be solved conveniently in the Fourier domain. Let

$$\tilde{\Sigma}(k_x,k_y,t)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\Sigma(x,y,t)\,e^{-ik_xx}\,e^{-ik_yy}\,dx\,dy,$$

etc., so that

$$\left(-k^2+\frac{\partial^2}{\partial z^2}\right)\tilde{\Phi}_{\mathrm{d}}=4\pi G\tilde{\Sigma}\,\delta(z),$$

where

$$k=\sqrt{k_x^2+k_y^2}$$

is the horizontal wavenumber. The relevant solution (decaying as $|z|\to\infty$) for $k\neq0$ is

$$\tilde{\Phi}_{\mathrm{d}}=-\frac{2\pi G\tilde{\Sigma}}{k}\,e^{-k|z|},$$

so that

$$\left[\frac{\partial\tilde{\Phi}_{\mathrm{d}}}{\partial z}\right]_{0-}^{0+}=4\pi G\tilde{\Sigma},$$

as required. So

$$\tilde{\Phi}_{\mathrm{d,m}}=-\frac{2\pi G\tilde{\Sigma}}{k}.$$

(The $k=0$ component of the potential gives no horizontal force anyway.)

11.2. Conservation of potential vorticity

Use the vector identity

$$(\boldsymbol{\nabla}\times\mathbf{u})\times\mathbf{u}=\mathbf{u}\cdot\boldsymbol{\nabla}\mathbf{u}-\boldsymbol{\nabla}\left(\tfrac{1}{2}|\mathbf{u}|^2\right)$$

to rewrite the equation of motion as

$$\frac{\partial\mathbf{u}}{\partial t}+[(2\boldsymbol{\Omega}+\boldsymbol{\nabla}\times\mathbf{u})\times\mathbf{u}]=-\boldsymbol{\nabla}(\cdots),$$

since $P=P(\Sigma)$. Take the curl:

$$\frac{\partial}{\partial t}(\boldsymbol{\nabla}\times\mathbf{u})+\boldsymbol{\nabla}\times[(2\boldsymbol{\Omega}+\boldsymbol{\nabla}\times\mathbf{u})\times\mathbf{u}]=\mathbf{0}.$$

Now use the vector identity

$$\boldsymbol{\nabla}\times(\mathbf{A}\times\mathbf{B})=\mathbf{B}\cdot\boldsymbol{\nabla}\mathbf{A}-\mathbf{A}\cdot\boldsymbol{\nabla}\mathbf{B}+\mathbf{A}(\boldsymbol{\nabla}\cdot\mathbf{B})-\mathbf{B}(\boldsymbol{\nabla}\cdot\mathbf{A})$$

to obtain (since the problem is 2D)

$$\begin{aligned}\left(\frac{\partial}{\partial t}+\mathbf{u}\cdot\boldsymbol{\nabla}\right)(2\boldsymbol{\Omega}+\boldsymbol{\nabla}\times\mathbf{u})&=-(2\boldsymbol{\Omega}+\boldsymbol{\nabla}\times\mathbf{u})(\boldsymbol{\nabla}\cdot\mathbf{u})\\&=(2\boldsymbol{\Omega}+\boldsymbol{\nabla}\times\mathbf{u})\frac{1}{\Sigma}\left(\frac{\partial}{\partial t}+\mathbf{u}\cdot\boldsymbol{\nabla}\right)\Sigma.\end{aligned}$$

Thus

$$\frac{Df}{Dt}=0,$$

where

$$f=\frac{2\Omega+(\boldsymbol{\nabla}\times\mathbf{u})_z}{\Sigma}$$

is the potential vorticity or 'vortensity' (vorticity divided by surface density).

Vortices and zonal flows correspond to coherent structures in this conserved quantity. Unlike the incompressible 2D case, though, vortex dynamics is not the whole story. Vortical disturbances are coupled to acoustic ones, so a vortex can excite waves.

11.3. Linear stability of a uniform 2D self-gravitating sheet

The uniform basic state of the sheet is the solution $\mathbf{u}=-Sx\,\mathbf{e}_y$, $\Sigma=\text{constant}$, $P=\text{constant}$.

The linearized equations for small perturbations $\mathbf{v}$, Σ', etc., are

$$\begin{aligned}&\left(\frac{\partial}{\partial t}-Sx\frac{\partial}{\partial y}\right)\Sigma'+\Sigma\,\boldsymbol{\nabla}\cdot\mathbf{v}=0,\\&\left(\frac{\partial}{\partial t}-Sx\frac{\partial}{\partial y}\right)\mathbf{v}-Sv_x\,\mathbf{e}_y+2\boldsymbol{\Omega}\times\mathbf{v}=-\boldsymbol{\nabla}\Phi'_{\mathrm{d,m}}-\frac{1}{\Sigma}\boldsymbol{\nabla}P',\\&\nabla^2\Phi'_{\mathrm{d}}=4\pi G\Sigma'\,\delta(z),\end{aligned}$$

with

$$P'=\frac{dP}{d\Sigma}\Sigma'=v_{\mathrm{s}}^2\Sigma',$$

where $v_{\mathrm{s}}=\text{constant}$ is the (adiabatic) sound speed of the basic state.

The solutions are shearing waves:

$$\Sigma'(\mathbf{x},t) = \mathrm{Re}\left\{\tilde{\Sigma}'(t)\,\exp\left[i\mathbf{k}(t)\cdot\mathbf{x}\right]\right\},$$

etc., satisfying the amplitude equations

$$\begin{aligned}
&\frac{d\tilde{\Sigma}'}{dt} + \Sigma\, i\mathbf{k}\cdot\tilde{\mathbf{v}} = 0,\\
&\frac{d\tilde{v}_x}{dt} - 2\Omega\tilde{v}_y = -ik_x\left(\tilde{\Phi}'_{\mathrm{d,m}} + v_\mathrm{s}^2\frac{\tilde{\Sigma}'}{\Sigma}\right),\\
&\frac{d\tilde{v}_y}{dt} + (2\Omega - S)\tilde{v}_x = -ik_y\left(\tilde{\Phi}'_{\mathrm{d,m}} + v_\mathrm{s}^2\frac{\tilde{\Sigma}'}{\Sigma}\right),\\
&\tilde{\Phi}'_{\mathrm{d,m}} = -\frac{2\pi G\tilde{\Sigma}'}{k}.
\end{aligned}$$

The potential vorticity perturbation

$$\tilde{f}' = \frac{ik_x\tilde{v}_y - ik_y\tilde{v}_x}{\Sigma} - \frac{(2\Omega - S)\tilde{\Sigma}'}{\Sigma^2}$$

satisfies $d\tilde{f}'/dt$, as expected (**exercise**).

Consider axisymmetric waves: $k_y = 0$, $k_x = \text{constant}$, $k = |k_x|$. Then the equations have constant coefficients, so we can assume the amplitudes are $\propto e^{-i\omega t}$:

$$\begin{aligned}
&-i\omega\tilde{\Sigma}' + \Sigma\, ik_x\tilde{v}_x = 0,\\
&-i\omega\tilde{v}_x - 2\Omega\tilde{v}_y = -ik_x\left(v_\mathrm{s}^2 - \frac{2\pi G\Sigma}{|k_x|}\right)\frac{\tilde{\Sigma}'}{\Sigma},\\
&-i\omega\tilde{v}_y + (2\Omega - S)\tilde{v}_x = 0.
\end{aligned}$$

Multiply the second equation by $i\omega$ and eliminate $\tilde{\Sigma}'$ and $\tilde{v}_y$:

$$\omega^2\tilde{v}_x - 2\Omega(2\Omega - S)\tilde{v}_x = k_x^2\left(v_\mathrm{s}^2 - \frac{2\pi G\Sigma}{|k_x|}\right)\tilde{v}_x.$$

We deduce the dispersion relation for *density waves*:

$$\omega^2 = \Omega_r^2 - 2\pi G\Sigma|k_x| + v_\mathrm{s}^2 k_x^2. \tag{1}$$

There is also a time-independent ($\omega = 0$) vortical solution ($\tilde{f}' \neq 0$) with $\tilde{v}_x = 0$. This involves a sinusoidal azimuthal velocity perturbation $v_y(x)$ giving rise to a Coriolis force that is balanced by pressure and self-gravity: an example of a zonal flow.

The dispersion relation (1) has positive, stabilizing contributions from inertial forces (Ω_r^2) and acoustic forces ($v_\mathrm{s}^2 k_x^2$), and a negative, destabilizing contribution from self-gravity. It describes 'acoustic–inertial waves' that can potentially be destabilized by self-gravity.

The disc is unstable to axisymmetric disturbances if $\omega^2 < 0$ for some k_x. ω^2 is minimized wrt $|k_x|$ when

$$0 = -2\pi G\Sigma + 2v_\mathrm{s}^2|k_x| \qquad \Rightarrow \qquad |k_x| = \frac{\pi G\Sigma}{v_\mathrm{s}^2},$$

so

$$(\omega^2)_\mathrm{min} = \Omega_r^2 - \frac{(\pi G\Sigma)^2}{v_\mathrm{s}^2} = \Omega_r^2\left(1 - \frac{1}{Q^2}\right),$$

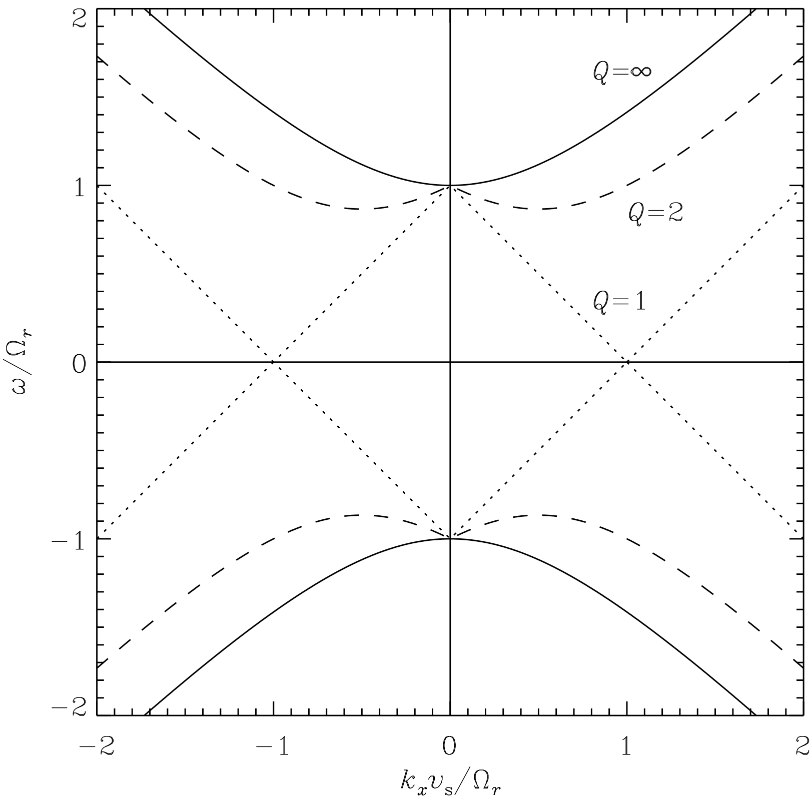
where

$$Q = \frac{v_\mathrm{s}\Omega_r}{\pi G\Sigma}$$

is the *Toomre stability parameter*. We have *gravitational instability* (GI) if $Q < 1$.

The definition of Q involves the product of the stabilizing influences (acoustic and inertial restoring forces) divided by a measure of the destabilizing influence (self-gravity).

Dispersion relations for density waves:



Lecture 12: Outcome of gravitational instability

12.1. Outcome of gravitational instability

If $Q < 1$, there is an axisymmetric GI that grows exponentially. The disc tends to break up into rings.

If $1 < Q \lesssim 2$, there can be a weaker, non-axisymmetric GI involving substantial transient growth. The disc tends to form spiral waves or clumps.

Since $Q \propto v_\mathrm{s} \propto \sqrt{T}$, thermostatic regulation is possible. If Q falls below about 2, instability occurs, producing motion that is dissipated through shocks and viscosity, heating the disc and raising Q.

Two possible outcomes of the GI are:

- *fragmentation*: formation of gravitationally bound objects (moonlets, planets, stars, etc.);
- *gravitational turbulence*: sustained activity of non-axisymmetric density waves while the disc maintains $1 < Q \lesssim 2$.

Efficient cooling promotes fragmentation.

The typical horizontal scale of the GI is a few times the scaleheight H. e.g. the fastest-growing mode of the axisymmetric GI has a radial wavelength $2\pi/k_x = 2\pi Q(v_\mathrm{s}/\Omega_r) \sim 2\pi Q\sqrt{\gamma}H$.

The ratio of the ('local') disc mass to the central mass for a Keplerian disc can be estimated as

$$\frac{\pi r^2\Sigma}{M} = \frac{1}{Q}\frac{v_\mathrm{s}\Omega r^2}{GM} = \frac{\sqrt{\gamma}}{Q}\frac{c_\mathrm{s}}{r\Omega} \sim \frac{H}{r}$$

when $Q \sim 1$. The typical mass of fragments formed by the GI is then a few times $(H/r)^3 M$. The GI can potentially form giant planets in the outer parts of sufficiently massive protoplanetary discs, or stars in the outer parts of discs in active galactic nuclei.

12.2. Thermal balance in a Keplerian disc with gravitational turbulence

Gravitational turbulence produces outward angular-momentum transport, which can be quantitifed by a dimensionless viscosity parameter α.

Suppose the disc has a constant cooling timescale $\tau = \beta/\Omega$, where β is dimensionless. The cooling rate per unit area is then

$$\mathscr{C} = \frac{P}{\gamma - 1}\frac{1}{\tau},$$

while the heating rate per unit area is

$$\mathscr{H} = \frac{9}{4}\alpha P\Omega.$$

Equating these gives a relation between α, β and γ (also valid for processes other than the GI):

$$\frac{9}{4}\alpha\beta(\gamma - 1) = 1.$$

Numerical simulations of the GI show fragmentation for $\beta \lesssim 4$ and turbulence for $\beta \gtrsim 4$, implying a maximum α of about $1/(9(\gamma - 1))$.

What is the mean effective viscosity $\bar{\nu}$ in a self-gravitating Keplerian disc with $Q \approx 1$?

$$Q = \frac{v_\mathrm{s}\Omega}{\pi G \Sigma} \qquad \Rightarrow \qquad \bar{\nu} = \frac{\alpha c_\mathrm{s}^2}{\Omega} = \frac{\alpha v_\mathrm{s}^2}{\gamma \Omega} = \frac{\alpha Q^2 (\pi G \Sigma)^2}{\gamma \Omega^3}.$$

If $\beta =$ constant then $\alpha =$ constant and $\bar{\nu} \propto \Sigma^2 r^{9/2}$.

Lecture 13: Magnetic fields in discs

13.1. Equations of magnetohydrodynamics (MHD)

To examine the role of magnetic fields in astrophysical discs, we simplify the problem by considering a homogeneous, incompressible, inviscid fluid of uniform density ρ and electrical conductivity σ. We work in rationalized units. To convert the magnetic field $\mathbf{B}$ from rationalized to Gaussian units, multiply by $\sqrt{\mu_0/4\pi}$.

The *induction equation*,

$$\frac{\partial \mathbf{B}}{\partial t} + \mathbf{u}\cdot\boldsymbol{\nabla}\mathbf{B} = \mathbf{B}\cdot\boldsymbol{\nabla}\mathbf{u} + \eta\nabla^2\mathbf{B},$$

describes the advection of the magnetic field $\mathbf{B}$ by the fluid flow, together with its diffusion due to resistivity. The *magnetic diffusivity* is

$$\eta = \frac{1}{\mu_0\sigma}.$$

The induction equation comes from Maxwell's equations without the displacement current,

$$\frac{\partial \mathbf{B}}{\partial t} = -\boldsymbol{\nabla}\times\mathbf{E}, \qquad \boldsymbol{\nabla}\cdot\mathbf{B} = 0, \qquad \boldsymbol{\nabla}\times\mathbf{B} = \mu_0\mathbf{J},$$

together with Ohm's Law for a moving medium,

$$\mathbf{J} = \sigma(\mathbf{E} + \mathbf{u}\times\mathbf{B}).$$

The *Lorentz force* per unit volume is the divergence of the *Maxwell stress tensor*

$$\mathsf{M} = \frac{\mathbf{B}\mathbf{B}}{\mu_0} - \frac{|\mathbf{B}|^2}{2\mu_0}\mathtt{I},$$

of which the first term represents a *magnetic tension* in the field lines and the second term represents an isotropic *magnetic pressure*. Since $\boldsymbol{\nabla}\cdot\mathbf{B} = 0$, we have

$$\boldsymbol{\nabla}\cdot\mathsf{M} = \frac{1}{\mu_0}\mathbf{B}\cdot\boldsymbol{\nabla}\mathbf{B} - \boldsymbol{\nabla}\left(\frac{|\mathbf{B}|^2}{2\mu_0}\right).$$

The equation of motion including the Lorentz force is

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}\cdot\boldsymbol{\nabla}\mathbf{u} = -\boldsymbol{\nabla}\Phi - \frac{1}{\rho}\boldsymbol{\nabla}\Pi + \frac{1}{\mu_0\rho}\mathbf{B}\cdot\boldsymbol{\nabla}\mathbf{B},$$

where

$$\Pi = p + \frac{|\mathbf{B}|^2}{2\mu_0}$$

is the total (gas plus magnetic) pressure.

The incompressibility and solenoidal conditions are

$$\boldsymbol{\nabla}\cdot\mathbf{u} = \boldsymbol{\nabla}\cdot\mathbf{B} = 0.$$

In the context of the local model of astrophysical discs, we add the Coriolis and centrifugal forces to obtain the equation of motion

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \boldsymbol{\nabla} \mathbf{u} + 2\boldsymbol{\Omega} \times \mathbf{u} = -\boldsymbol{\nabla} \Phi_{\mathrm{t}} - \frac{1}{\rho} \boldsymbol{\nabla} \Pi + \frac{1}{\mu_0 \rho} \mathbf{B} \cdot \boldsymbol{\nabla} \mathbf{B}.$$

Rotation does not affect the induction equation.

13.2. Horizontally invariant solutions

Write $\mathbf{u} = -Sx\,\mathbf{e}_y + \mathbf{v}$, where $\mathbf{v}$ is the departure from orbital motion. Look for horizontally invariant solutions in which $\mathbf{v}$, $\mathbf{B}$ and Π are independent of x and y. Then $\partial v_z/\partial z = \partial B_z/\partial z = 0$ and

$$\begin{aligned}
\frac{Dv_x}{Dt} - 2\Omega v_y &= \frac{B_z}{\mu_0 \rho} \frac{\partial B_x}{\partial z}, \\
\frac{Dv_y}{Dt} + (2\Omega - S) v_x &= \frac{B_z}{\mu_0 \rho} \frac{\partial B_y}{\partial z}, \\
\frac{\partial v_z}{\partial t} &= -\Omega_z^2 z - \frac{1}{\rho} \frac{\partial \Pi}{\partial z}, \\
\frac{DB_x}{Dt} &= B_z \frac{\partial v_x}{\partial z} + \eta \frac{\partial^2 B_x}{\partial z^2}, \\
\frac{DB_y}{Dt} &= -SB_x + B_z \frac{\partial v_y}{\partial z} + \eta \frac{\partial^2 B_y}{\partial z^2}, \\
\frac{\partial B_z}{\partial t} &= 0,
\end{aligned}$$

with

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + v_z \frac{\partial}{\partial z}.$$

In this model, $B_z = \text{constant}$ is a conserved uniform vertical magnetic flux passing through the disc.

Assume that the disc has upper and lower surfaces $z = z_{\pm}(t)$, with full thickness $z^+ - z^- = \Sigma/\rho = \text{constant}$. Above and below the disc, assume that $\rho = 0$ and magnetic field is force-free: $B_x = B_x^{\pm} = \text{constant}$ and $B_y = B_y^{\pm}$, respectively. The inclinations $B_{x,y}^{\pm}/B_z$ are determined by global considerations.

Exercise: Integrate the vertical equation of motion over the disc, with the boundary condition $p = 0$ at $z = z^{\pm}$, to obtain

$$\ddot{Z} = -\Omega_z^2 Z - \frac{(\Pi^+ - \Pi^-)}{\Sigma},$$

where $Z = (z^+ + z^-)/2$ is the height of the centre of mass and $\Pi^{\pm} = |\mathbf{B}^+|^2/2\mu_0$. This equation allows a free oscillation about an equilibrium position (which is $Z = 0$ if $\Pi^+ = \Pi^-$).

Assume that the vertical oscillation is absent so that $v_z = 0$. Then the equations of the model reduce to a linear problem:

$$\begin{aligned}
\frac{\partial v_x}{\partial t} - 2\Omega v_y &= \frac{B_z}{\mu_0 \rho}\frac{\partial B_x}{\partial z}, \\
\frac{\partial v_y}{\partial t} + (2\Omega - S)v_x &= \frac{B_z}{\mu_0 \rho}\frac{\partial B_y}{\partial z}, \\
\frac{\partial B_x}{\partial t} &= B_z\frac{\partial v_x}{\partial z} + \eta\frac{\partial^2 B_x}{\partial z^2}, \\
\frac{\partial B_y}{\partial t} + SB_x &= B_z\frac{\partial v_y}{\partial z} + \eta\frac{\partial^2 B_y}{\partial z^2}.
\end{aligned}$$

13.3. Equilibrium solutions

In a steady state

$$\begin{aligned}
-2\Omega v_y &= \frac{B_z}{\mu_0 \rho}\frac{dB_x}{dz}, \\
(2\Omega - S)v_x &= \frac{B_z}{\mu_0 \rho}\frac{dB_y}{dz}, \\
0 &= B_z\frac{dv_x}{dz} + \eta\frac{d^2 B_x}{dz^2}, \\
SB_x &= B_z\frac{dv_y}{dz} + \eta\frac{d^2 B_y}{dz^2}.
\end{aligned}$$

Eliminate variables in favour of B_x to obtain (**exercise**)

$$\frac{d^2 B_x}{dz^2} + K^2 B_x = 0, \qquad K^2 = \frac{2\Omega S v_{\mathrm{az}}^2}{v_{\mathrm{az}}^4 + \eta^2\Omega_r^2},$$

where

$$v_{\mathrm{az}} = \frac{B_z}{\sqrt{\mu_0 \rho}}$$

is the vertical *Alfvén velocity*. The solution is a combination of $\sin(Kz)$ and $\cos(Kz)$.

The situation that is usually considered involves symmetrical boundary conditions: $B_x = \pm B_x^+$ and $B_y = \pm B_y^+$ at $z = \pm z^+$. The relevant solution is

$$\begin{aligned}
B_x &= B_x^+ \frac{\sin(Kz)}{\sin(Kz^+)}, \\
B_y &= B_y^+ \frac{z}{z^+} - \frac{(2\Omega - S)\eta}{v_{\mathrm{az}}^2} B_x^+ \left[\frac{\sin(Kz)}{\sin(Kz^+)} - \frac{z}{z^+}\right], \\
v_x &= \frac{v_{\mathrm{az}}^2}{(2\Omega - S)z^+}\frac{B_y^+}{B_z} - \eta\frac{B_x^+}{B_z}\left[\frac{K\cos(Kz)}{\sin(Kz^+)} - \frac{1}{z^+}\right], \\
v_y &= -\frac{v_{\mathrm{az}}^2}{2\Omega}\frac{B_x^+}{B_z}\frac{K\cos(Kz)}{\sin(Kz^+)}.
\end{aligned}$$

The meridional (x and z) components of $\mathbf{B}$ are called the *poloidal magnetic field*, while the azimuthal (y) component is called the *toroidal magnetic field*.

The poloidal magnetic field bends in the xz plane to match the boundary conditions. The shape is close to a parabola if $Kz^+ \ll 1$, i.e. in the limit of a strong field and/or a high resistivity; otherwise the shape is more 'bendy'.

There is a z-dependent departure from the orbital motion (v_y) with the Coriolis and Lorentz forces balancing. We have *isorotation* (i.e. constant angular velocity) along field lines if $\eta = 0$. More generally, the ratio of the two terms on the RHS of the y-component of the induction equation is $\eta^2\Omega_r^2/v_{\mathrm{a}z}^4$.

There is a mean radial velocity (i.e. an accretion flow) if a non-zero magnetic torque $\propto B_y^+ B_z$ acts on the disc to remove (or add) angular momentum. (This could result either from a magnetized outflow or from a magnetic connection to an external object rotating at a different rate.)

Above the disc, assuming very low density, we have

$$B_x = B_x^+, \qquad B_y = B_y^+, \qquad v_x = \text{constant}, \qquad v_y = \frac{B_x^+}{B_z} S z + \text{constant}.$$

The uniform, force-free magnetic field acts as a rigid channel for the gas. The net acceleration parallel to $\mathbf{B}$ due to inertial forces and gravity is proportional to

$$\frac{B_x^+}{B_z} 2\Omega v_y - \frac{B_y^+}{B_z}(2\Omega - S)v_x - \Omega_z^2 z = \left(\frac{B_x^+}{B_z}\right)^2 2\Omega S z - \Omega_z^2 z + \text{constant}.$$

If the field is sufficiently inclined to the vertical, i.e. if

$$\left(\frac{B_x^+}{B_z}\right)^2 > \frac{\Omega_z^2}{2\Omega S},$$

then this net acceleration increases with z and will become positive at some height above the disc. A hydrostatic solution is then impossible and an outflow (jet or wind) is launched along the field lines. This is known as *magnetocentrifugal acceleration*. For a Keplerian disc ($\Omega_z = \Omega$, $S/\Omega = 3/2$), it requires $i > 30°$, where $i = \arctan|B_x^+/B_z|$ is the inclination.

Lecture 14: The magnetorotational instability

14.1. Stability analysis

We return to the equations

$$\begin{aligned}\frac{\partial v_x}{\partial t}-2\Omega v_y&=\frac{B_z}{\mu_0\rho}\frac{\partial B_x}{\partial z},\\ \frac{\partial v_y}{\partial t}+(2\Omega-S)v_x&=\frac{B_z}{\mu_0\rho}\frac{\partial B_y}{\partial z},\\ \frac{\partial B_x}{\partial t}&=B_z\frac{\partial v_x}{\partial z}+\eta\frac{\partial^2 B_x}{\partial z^2},\\ \frac{\partial B_y}{\partial t}+SB_x&=B_z\frac{\partial v_y}{\partial z}+\eta\frac{\partial^2 B_y}{\partial z^2}\end{aligned}$$

describing the time-dependent vertical structure of a magnetized disc.

Consider perturbations to an equilibrium state, of the form

$$\delta v_x=\mathrm{Re}\left(\tilde{v}_x\,e^{\lambda t+ikz}\right),$$

etc., where λ is the (possibly complex) growth rate and k is the (real) vertical wavenumber. The equations require

$$\begin{aligned}\lambda\tilde{v}_x-2\Omega\tilde{v}_y&=\frac{ikB_z}{\mu_0\rho}\tilde{B}_x,\\ \lambda\tilde{v}_y+(2\Omega-S)\tilde{v}_x&=\frac{ikB_z}{\mu_0\rho}\tilde{B}_y,\\ (\lambda+\eta k^2)\tilde{B}_x&=ikB_z\tilde{v}_x,\\ (\lambda+\eta k^2)\tilde{B}_y+S\tilde{B}_x&=ikB_z\tilde{v}_y.\end{aligned}$$

Multiply the first two equations by ikB_z and use the last two to substitute for $\tilde{v}_x$ and $\tilde{v}_y$:

$$\begin{aligned}\lambda\lambda_\eta\tilde{B}_x-2\Omega\left(\lambda_\eta\tilde{B}_y+S\tilde{B}_x\right)&=-\omega_{\mathrm{a}}^2\tilde{B}_x,\\ \lambda\left(\lambda_\eta\tilde{B}_y+S\tilde{B}_x\right)+(2\Omega-S)\lambda_\eta\tilde{B}_x&=-\omega_{\mathrm{a}}^2\tilde{B}_y,\end{aligned}$$

where $\lambda_\eta=\lambda+\eta k^2$ and the *Alfvén frequency* is

$$\omega_{\mathrm{a}}=\mathbf{k}\cdot\mathbf{v}_{\mathrm{a}}=\frac{kB_z}{\sqrt{\mu_0\rho}}.$$

Algebraic elimination leads to the *magnetorotational dispersion relation*

$$(\lambda\lambda_\eta+\omega_{\mathrm{a}}^2)^2+\Omega_r^2\lambda_\eta^2-2\Omega S\omega_{\mathrm{a}}^2=0.$$

This gives marginal stability ($\lambda=0$) for $k^2=K^2$ (K being the equilibrium wavenumber) and instability for $k^2<K^2$. To prove this we use the *Routh–Hurwitz stability criteria*: the roots of the real quartic polynomial

$$\lambda^4+a\lambda^3+b\lambda^2+c\lambda+d=0$$

all have $\mathrm{Re}(\lambda) < 0$ if and only if

$$a, b, c, d > 0 \qquad \text{and} \qquad abc - a^2 d - c^2 > 0.$$

In our case we have

$$a = 2\eta k^2, \qquad b = 2\omega_\mathrm{a}^2 + (\eta k^2)^2 + \Omega_r^2, \qquad c = 2(\omega_\mathrm{a}^2 + \Omega_r^2)\eta k^2, \qquad d = \omega_\mathrm{a}^4 + \Omega_r^2(\eta k^2)^2 - 2\Omega S\omega_\mathrm{a}^2,$$

giving (**exercise**)

$$abc - a^2 d - c^2 = (2\eta k^2)^2 \omega_\mathrm{a}^2 \left[(\eta k^2)^2 + 4\Omega^2\right] > 0.$$

Assume that $\Omega_r^2 > 0$; otherwise we already have orbital and hydrodynamic instability. Then all of the Routh–Hurwitz stability criteria are satisfied, except possibly the criterion $d > 0$. But

$$\begin{aligned} d &= \left(v_{\mathrm{a}z}^4 + \eta^2\Omega_r^2\right) k^4 - 2\Omega S v_{\mathrm{a}z}^2 k^2 \\ &= \left(v_{\mathrm{a}z}^4 + \eta^2\Omega_r^2\right) k^2(k^2 - K^2), \end{aligned}$$

and we have *magnetorotational instability* (MRI) for $d < 0$.

The usual boundary conditions for normal modes are that $\delta B_x = \delta B_y = 0$ at $z = \pm z^+$. This gives solutions involving $\sin(kz)$ or $\cos(kz)$ (note that the dispersion relation is even in k, so $e^{\pm ikz}$ can be combined), with the quantization

$$k = \frac{n\pi}{2z^+}, \qquad n = 1, 2, 3, \dots.$$

The $n = 1$ mode has the lowest value of k and therefore determines the overall stability of the equilibrium, although it may not be the fastest-growing mode.

14.2. The ideal MRI

For ideal MHD (the perfectly conducting limit of zero resistivity) we set $\eta = 0$, in which case $\lambda_\eta = \lambda$. The dispersion relation is then a quadratic for λ^2:

$$\lambda^4 + (2\omega_\mathrm{a}^2 + \Omega_r^2)\lambda^2 + \omega_\mathrm{a}^2(\omega_\mathrm{a}^2 - 2\Omega S) = 0,$$

with discriminant

$$\begin{aligned} (2\omega_\mathrm{a}^2 + \Omega_r^2)^2 - 4\omega_\mathrm{a}^2(\omega_\mathrm{a}^2 - 2\Omega S) &= \Omega_r^4 + 4\omega_\mathrm{a}^2(\Omega_r^2 + 2\Omega S) \\ &= \Omega_r^4 + 16\omega_\mathrm{a}^2\Omega^2 \end{aligned}$$

and roots

$$\lambda^2 = -\omega_\mathrm{a}^2 + \frac{1}{2}\left(-\Omega_r^2 \pm \sqrt{\Omega_r^4 + 16\omega_\mathrm{a}^2\Omega^2}\right).$$

Assume again that $\Omega_r^2 > 0$. The $+$ root is maximized wrt ω_a^2 when

$$0 = \frac{\partial\lambda^2}{\partial\omega_\mathrm{a}^2} = -1 + \frac{4\Omega^2}{\sqrt{\cdots}}$$

$$\sqrt{\cdots} = 4\Omega^2$$

$$\Omega_r^4 + 16\omega_\mathrm{a}^2\Omega^2 = 16\Omega^4$$

$$\omega_\mathrm{a}^2 = \Omega^2 - \frac{\Omega_r^4}{16\Omega^2},$$

giving

$$
\begin{aligned}
(\lambda^2)_{\max} &= -\Omega^2 + \frac{\Omega_r^4}{16\Omega^2} + \frac{1}{2}(-\Omega_r^2 + 4\Omega^2) \\
&= \Omega^2 \left(1 - \frac{\Omega_r^2}{4\Omega^2}\right)^2 \\
&= \Omega^2 \left(\frac{2\Omega S}{4\Omega^2}\right)^2 \\
&= \left(\frac{S}{2}\right)^2 .
\end{aligned}
$$

So the maximum growth rate is half the orbital shear rate, *independent of the magnetic field*. The weaker the field is, the shorter the wavelength of the fastest-growing mode, to achieve $\omega_{\mathrm{a}} \sim \Omega$.

Exercise: Show that the fastest-growing mode has $\delta v_x = \delta v_y$ and $\delta B_x = -\delta B_y$, which maximizes the correlations leading to outward angular momentum transport:

$$
-T_{xy} = \rho\, \delta v_x\, \delta v_y - \frac{\delta B_x\, \delta B_y}{\mu_0}.
$$

This shear stress also extracts energy from the orbital shear, allowing the perturbation to grow.

For ideal instability we require

$$
\omega_{\mathrm{a}}^2(\omega_{\mathrm{a}}^2 - 2\Omega S) < 0.
$$

Crucially, this condition is satisfied for a Keplerian disc, provided that the field is not too strong, whereas the Rayleigh criterion for hydrodynamic instability, $4\Omega^2 - 2\Omega S < 0$, is not.

The $n = 1$ mode, with wavenumber $k = \pi/2z^+$, is the last to be stabilized as B_z is increased. The disc is unstable for

$$
0 < \frac{v_{\mathrm{a}z}}{\Omega z^+} < \frac{2\sqrt{2q}}{\pi} \qquad (\approx 1.1 \text{ for a Keplerian disc }).
$$

14.3. Nonlinear outcome

The MRI typically develops into sustained MHD turbulence in discs that are sufficiently ionized (η small enough) and not very strongly magnetized. It leads to outward angular-momentum transport with typically $\alpha \lesssim 0.1$, depending on the field strength and the degree of ionization.

In the absence of a large-scale, imposed magnetic field, it is thought that the MRI can act as a *dynamo*, in which the turbulent motions due to the instability sustain the magnetic field against Ohmic dissipation. Whether this dynamo can operate at the very low viscosities found in astrophysical discs is an open question.

Lecture 15: Satellite–disc interaction

The interaction of an orbiting companion (moon, planet, star, black hole, etc.) with a disc is one of the most important problems in the theory of astrophysical discs. For such massive satellites (as opposed to dust grains or planetesimals) the interaction is predominantly gravitational rather than hydrodynamic. The gravity of the satellite perturbs nearby orbital motion in the disc and excites waves and other disturbances. Angular momentum and energy are exchanged, leading to orbital evolution of the satellite, e.g. inward or outward radial migration of a planet in a protoplanetary disc.

15.1. Excitation of epicyclic motion by a satellite

We consider the dynamics of test particles in the xy plane, using the local approximation:

$$\begin{aligned}\ddot{x} - 2\Omega\dot{y} &= 2\Omega S x - \frac{\partial\Psi}{\partial x},\\ \ddot{y} + 2\Omega\dot{x} &= -\frac{\partial\Psi}{\partial y}.\end{aligned}$$

For a satellite of mass M_s on a circular orbit at the reference radius ($x_\mathrm{s} = y_\mathrm{s} = 0$), the potential of the satellite is

$$\Psi = -\frac{GM_\mathrm{s}}{\sqrt{x^2+y^2}}.$$

The general solution in the absence of Ψ (recall §3.1) is

$$\begin{aligned}x &= x_0 + \mathrm{Re}\left(A\,e^{-i\Omega_r t}\right),\\ y &= y_0 - Sx_0 t + \mathrm{Re}\left(\frac{2\Omega A}{i\Omega_r}\,e^{-i\Omega_r t}\right).\end{aligned}$$

This involves an epicyclic oscillation of complex amplitude A around a guiding centre that follows a circular orbit $(x_0, y_0 - Sx_0 t)$.

To express x_0 and A in terms of position and velocity:

$$\begin{aligned}x &= x_0 + \mathrm{Re}\left(A\,e^{-i\Omega_r t}\right),\\ \dot{x} &= \mathrm{Re}\left(-i\Omega_r A\,e^{-i\Omega_r t}\right) = \Omega_r\,\mathrm{Im}\left(A\,e^{-i\Omega_r t}\right),\\ \dot{y} &= -Sx_0 - 2\Omega\,\mathrm{Re}\left(A\,e^{-i\Omega_r t}\right).\end{aligned}$$

So the canonical y-momentum (per unit mass) is

$$p_y = \dot{y} + 2\Omega x = (2\Omega - S)x_0 = \frac{\Omega_r^2}{2\Omega}x_0$$

and we can find the epicyclic amplitude from

$$\begin{aligned}A\,e^{-i\Omega_r t} &= \mathrm{Re}\left(A\,e^{-i\Omega_r t}\right) + i\,\mathrm{Im}\left(A\,e^{-i\Omega_r t}\right)\\ &= -\frac{(\dot{y}+Sx)}{(2\Omega - S)} + \frac{i\dot{x}}{\Omega_r}\\ A &= \left[-\frac{2\Omega}{\Omega_r^2}(\dot{y}+Sx) + \frac{i\dot{x}}{\Omega_r}\right]e^{i\Omega_r t}.\end{aligned}$$

The specific energy is

$$\varepsilon = \frac{1}{2}\left(\dot{x}^2 + \dot{y}^2\right) - \Omega S x^2.$$

Now

$$\begin{aligned}
\Omega_r A &= \left[-\frac{2\Omega}{\Omega_r}(\dot{y} + Sx) + i\dot{x}\right] e^{i\Omega_r t} \\
\Omega_r^2 |A|^2 &= \dot{x}^2 + \frac{4\Omega^2}{\Omega_r^2}(\dot{y} + Sx)^2 \\
&= 2\varepsilon - \dot{y}^2 + 2\Omega S x^2 + \frac{4\Omega^2}{\Omega_r^2}(\dot{y} + Sx)^2 \\
&= 2\varepsilon + \frac{2\Omega S}{\Omega_r^2}\dot{y}^2 + \frac{8\Omega^2 S}{\Omega_r^2}\dot{y}x + \frac{8\Omega^3 S}{\Omega_r^2}x^2 \\
&= 2\varepsilon + \frac{2\Omega S}{\Omega_r^2}(\dot{y} + 2\Omega x)^2,
\end{aligned}$$

so

$$\varepsilon = \frac{1}{2}\Omega_r^2 |A|^2 - \frac{\Omega S}{\Omega_r^2} p_y^2 = \text{constant}.$$

In the presence of a satellite potential, we have instead

$$\begin{aligned}
\dot{p}_y &= -\frac{\partial \Psi}{\partial y}, \\
\varepsilon + \Psi &= \text{constant}, \\
\dot{A} &= \left[-\frac{2\Omega}{\Omega_r^2}(\ddot{y} + S\dot{x}) + \frac{i\ddot{x}}{\Omega_r} - \frac{2i\Omega}{\Omega_r}(\dot{y} + Sx) - \dot{x}\right] e^{i\Omega_r t}, \\
&= \left[-\frac{2\Omega}{\Omega_r^2}(\ddot{y} + 2\Omega\dot{x}) + \frac{i}{\Omega_r}(\ddot{x} - 2\Omega\dot{y} - 2\Omega S x)\right] e^{i\Omega_r t}, \\
&= \left(\frac{2\Omega}{\Omega_r^2}\frac{\partial \Psi}{\partial y} - \frac{i}{\Omega_r}\frac{\partial \Psi}{\partial x}\right) e^{i\Omega_r t}.
\end{aligned}$$

15.2. Linear perturbation theory

We can calculate the change ΔA induced by Ψ using linear perturbation theory.

The basic state is an unperturbed circular orbit ($A = 0$) at radial separation x_0 from the satellite:

$$x = x_0 = \text{constant}, \qquad y = -Sx_0 t.$$

Then, with $\Psi = -GM_\mathrm{s}(x^2 + y^2)^{-1/2}$,

$$\begin{aligned}
\dot{A} &= \left(\frac{2\Omega}{\Omega_r^2}\frac{\partial \Psi}{\partial y} - \frac{i}{\Omega_r}\frac{\partial \Psi}{\partial x}\right) e^{i\Omega_r t} \\
&= GM_\mathrm{s}(x^2 + y^2)^{-3/2}\left(\frac{2\Omega y}{\Omega_r^2} - \frac{ix}{\Omega_r}\right) e^{i\Omega_r t} \\
&\approx -i\frac{GM_\mathrm{s}}{\Omega_r x_0^2}(1 + S^2 t^2)^{-3/2}\left(1 - i\frac{2\Omega}{\Omega_r}St\right) e^{i\Omega_r t},
\end{aligned}$$

giving (the real part of the integral vanishes by symmetry)

$$\begin{aligned}
\Delta A &= \int_{-\infty}^{\infty} \dot{A}\,dt \\
&= -i\frac{GM_{\rm s}}{\Omega_r x_0^2}\int_{-\infty}^{\infty}(1+S^2t^2)^{-3/2}\left(\cos\Omega_r t + \frac{2\Omega}{\Omega_r}St\sin\Omega_r t\right)\,dt.
\end{aligned}$$

Let

$$f(k) = \int_{-\infty}^{\infty}(1+x^2)^{-3/2}\cos kx\,dx = 2kK_1(k) \qquad (k>0),$$

where K_1 is a modified Bessel function. [$f(k)$ decreases monotonically from 2 to 0 as k increases from 0 to ∞.] Then

$$\Delta A = -iC\frac{GM_{\rm s}}{\Omega_r S x_0^2},$$

where

$$C = f\left(\frac{\Omega_r}{S}\right) - \frac{2\Omega}{\Omega_r}f'\left(\frac{\Omega_r}{S}\right)$$

is a function of q only. For Keplerian orbits ($\Omega_r/S = 2/3$), $C \approx 3.36$.

So the gravitational encounter of a test particle with the satellite excites an epicyclic oscillation at first order.

Long before and after the encounter, $\Psi \to 0$. Since $\varepsilon + \Psi$ is exactly conserved, $\Delta\varepsilon = 0$ in the encounter. But

$$\varepsilon = \frac{1}{2}\Omega_r^2|A|^2 - \frac{\Omega S}{\Omega_r^2}p_y^2,$$

so

$$\Delta(p_y^2) = \frac{\Omega_r^4}{2\Omega S}\Delta(|A|^2).$$

Assume a circular orbit before the encounter:

$$A = 0, \qquad p_y = \frac{\Omega_r^2}{2\Omega}x_0.$$

Then, after the encounter,

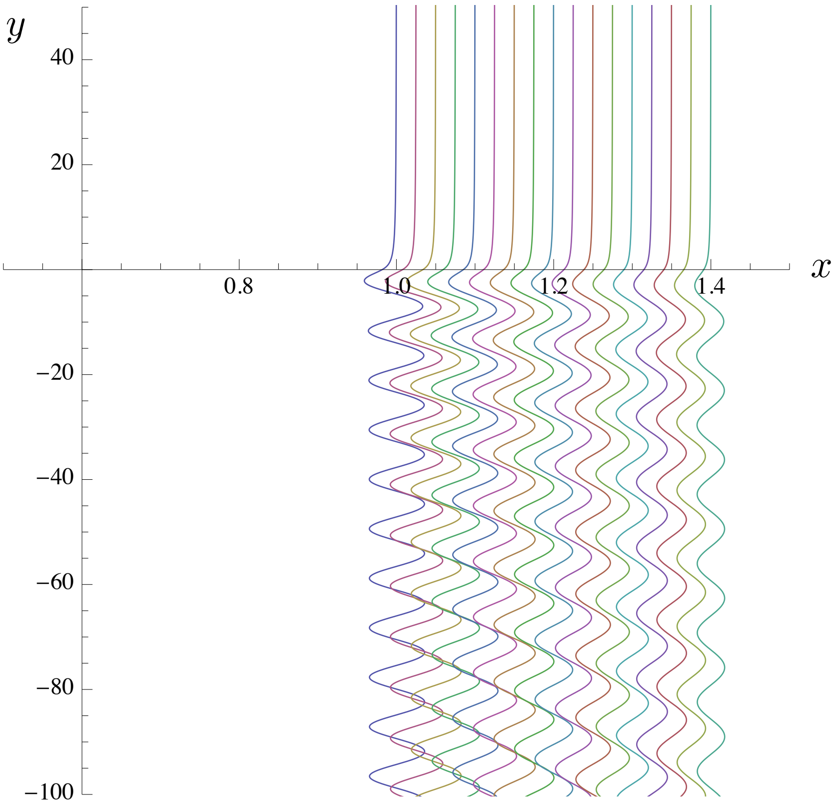
$$\begin{aligned}
A &\approx -iC\frac{GM_{\rm s}}{\Omega_r S x_0^2}, \\
\Delta(p_y^2) &\approx 2\frac{\Omega_r^2}{2\Omega}x_0\,\Delta p_y = \frac{\Omega_r^4}{2\Omega S}\left(C\frac{GM_{\rm s}}{\Omega_r S x_0^2}\right)^2,
\end{aligned}$$

so

$$\Delta p_y = \frac{(CGM_{\rm s})^2}{2S^3x_0^5},$$

correct to second order.

Some irreversibility or dissipation is implicit in assuming the initial orbit to be circular.



15.3. Impulse approximation

A simplified version of the calculation treats the interaction of the test particle with the satellite as a impulsive two-body interaction at the point of closest approach.

[FIGURE]

We estimate

$$
\begin{aligned}
\Delta v_\perp &\approx \frac{GM_\mathrm{s}}{x_0^2}\frac{1}{S} && \text{(acceleration} \times \text{time)} \\
\Delta(v_\perp^2) + \Delta(v_\parallel^2) &= 0 && \text{(conservation of energy)} \\
\left(\frac{GM_\mathrm{s}}{Sx_0^2}\right)^2 &+ 2Sx_0\,\Delta v_\parallel \approx 0 \\
\Delta v_\parallel &\approx -\frac{(GM_\mathrm{s})^2}{2S^3x_0^5},
\end{aligned}
$$

which is the same result but lacking the dimensionless factor $C^2 \approx 11.3$.

Lecture 16: Satellite–disc / Particle–disc interaction

16.1. Satellite–disc torques

The y component of the force on the disc, per unit x, at location x, is

$$\begin{aligned}
&\Delta p_y \quad \times \quad \text{surface density} \quad \times \quad \text{encounter rate} \\
&= \frac{(CGM_\mathrm{s})^2}{2S^3x^5} \cdot \Sigma \cdot |Sx| \\
&\propto x^{-4}\,\mathrm{sgn}(x).
\end{aligned}$$

The torque per unit radius is the same $\times r_0$. The satellite experiences an equal and opposite torque.

- The effect is of second order in M_s
- A similar result is obtained for the response of a fluid disc, in which case density waves are excited
- The x^{-4} divergence is moderated within $|x| \lesssim H$
- Angular-momentum transport is outward
- The gravitational satellite–disc interaction is effectively 'repulsive'!
- The one-sided torque leads to gap opening if M_s is large enough and ν small enough
- Asymmetries between inner and outer torques lead to a net torque on the satellite and to migration (usually inwards)

16.2. Drag forces on particles

The behaviour of solid particles such as dust grains in a gaseous disc is of fundamental importance in the processes of planet formation. Observations of discs around young stars often reveal more about the dust than the gas, even though the dust may constitute only about 1% of the total mass.

A particle of mass m with velocity $\dot{\mathbf{x}}$ moving in a gas with velocity $\mathbf{u}(\mathbf{x}, t)$ experiences a relative wind velocity $\mathbf{u} - \dot{\mathbf{x}}$ and a drag force

$$\mathbf{F} = k(\mathbf{u} - \dot{\mathbf{x}}).$$

For subsonic relative motion ($|\mathbf{u} - \dot{\mathbf{x}}| \ll v_{\mathrm{s}}$), the coefficient k can be regarded as independent of the relative velocity if either

- the size of the particle is small compared to the mean free path of the gas (the kinetic regime, *Epstein drag*), or
- the Reynolds number of the relative motion is small, resulting in a laminar fluid flow around it (the laminar hydrodynamic regime, *Stokes drag*).

The acceleration of the particle due to drag can then be written as

$$\frac{\mathbf{F}}{m} = \gamma(\mathbf{u} - \dot{\mathbf{x}}) = \frac{\mathbf{u} - \dot{\mathbf{x}}}{t_{\mathrm{s}}},$$

where $t_{\mathrm{s}} = m/k$ is the *stopping time* and $\gamma = 1/t_{\mathrm{s}}$. The stopping time is an increasing function of particle size.

(For larger particles and faster relative motion, k is an increasing function of the relative speed.)

16.3. Radial drift

In the local approximation, the equation of motion of a solid particle is

$$\begin{aligned}
\ddot{x} - 2\Omega\dot{y} &= 2\Omega S x + \gamma(u_x - \dot{x}), \\
\ddot{y} + 2\Omega\dot{x} &= \gamma(u_y - \dot{y}), \\
\ddot{z} &= -\Omega_z^2 z + \gamma(u_z - \dot{z}).
\end{aligned}$$

If we take the gas velocity to be the orbital shear flow $\mathbf{u} = -Sx\,\mathbf{e}_y$, then the equations are linear. Both the horizontal (epicyclic) and vertical oscillations of the particle are damped.

Exercise: Show that the general solution of these equations under this assumption involves an epicyclic oscillation with an amplitude that decays $\propto e^{-\gamma t}$, while the vertical motion behaves as a damped harmonic oscillator. Therefore particles tend to settle into circular orbits in the midplane.

Now allow for a small departure from the orbital shear flow, so that

$$\mathbf{u} = [-Sx + v_y(x)]\,\mathbf{e}_y.$$

The radial force balance for the gas associates the zonal flow $v_y(x)$ with a radial pressure gradient. In a 2D model,

$$-2\Omega v_y = -\frac{1}{\Sigma}\frac{\partial P}{\partial x}.$$

Write the particle velocity as

$$\dot{\mathbf{x}} = -Sx\,\mathbf{e}_y + \mathbf{w}.$$

Then the equation of motion of the particle is

$$\begin{aligned}
\dot{w}_x - 2\Omega w_y &= \gamma(-w_x),\\
\dot{w}_y + (2\Omega - S)w_x &= \gamma(v_y - w_y),\\
\dot{w}_z &= -\Omega_z^2 z + \gamma(-w_z).
\end{aligned}$$

To the extent that v_y can be treated as a constant, a steady solution for the dust velocity is given by $w_z = z = 0$ and

$$-2\Omega w_y = -\gamma w_x, \qquad (2\Omega - S)w_x = \gamma(v_y - w_y),$$

which gives

$$w_x = \left(\frac{2\Omega\gamma}{\gamma^2 + \Omega_r^2}\right) v_y, \qquad w_y = \left(\frac{\gamma^2}{\gamma^2 + \Omega_r^2}\right) v_y.$$

In terms of the dimensionless *Stokes number*

$$\mathrm{St} = \Omega_r t_\mathrm{s} = \frac{\Omega_r}{\gamma},$$

we have

$$w_x = \left(\frac{\mathrm{St}}{1 + \mathrm{St}^2}\right)\frac{1}{\Omega_r}\frac{1}{\Sigma}\frac{\partial P}{\partial x}, \qquad w_y = \frac{v_y}{1 + \mathrm{St}^2}.$$

Particles with $\mathrm{St} \ll 1$ (typically sub-cm grains) nearly follow the azimuthal motion of the gas, while larger particles with $\mathrm{St} \gg 1$ move independently. In either case the drag causes a radial drift up the pressure gradient. The drift speed is maximized for $\mathrm{St} = 1$.

In a featureless protoplanetary disc model in which P is a monotonically decreasing function of r, particles with $\mathrm{St} \sim 1$ drift into the central star on a timescale $\sim (r/H)^2$ times the orbital timescale, i.e. less than 1000 yr from 1 AU:

$$|w_x| \sim \frac{1}{r\Omega}\frac{P}{\Sigma} \sim \frac{c_\mathrm{s}^2}{r\Omega}$$

$$\frac{r}{|w_x|} \sim \frac{r^2\Omega}{H^2\Omega^2} \sim \left(\frac{r}{H}\right)^2 \frac{1}{\Omega}.$$

If the disc has structures such as zonal flows, vortices or spiral density waves in which local pressure maxima occur, then particles may be trapped in them, enhancing the processes of planet formation.

16.4. Trapping of dust in a vortex

Recall the flow inside an equilibrium elliptical vortex patch of aspect ratio r and strength ζ_0 (see §10.4):

$$\mathbf{u} = A\left(\frac{y}{r}, -rx\right), \qquad A = \frac{S}{r-1}, \qquad \frac{\zeta_0}{S} = -\frac{(r+1)}{r(r-1)}.$$

The equation of motion of a particle inside the vortex is

$$\begin{aligned}\ddot{x} - 2\Omega\dot{y} &= 2\Omega S x - \gamma(\dot{x} - u_x),\\ \ddot{y} + 2\Omega\dot{x} &= -\gamma(\dot{y} - u_y).\end{aligned}$$

This is a linear system with solutions $x, y \propto e^{\lambda t}$:

$$\begin{pmatrix}\lambda^2 + \gamma\lambda - 2\Omega S & -2\Omega\lambda - \gamma A r^{-1}\\ 2\Omega\lambda + \gamma A r & \lambda^2 + \gamma\lambda\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix} = \begin{pmatrix}0\\ 0\end{pmatrix}$$

$$\left(\lambda^2 + \gamma\lambda - 2\Omega S\right)\left(\lambda^2 + \gamma\lambda\right) + \left(2\Omega\lambda + \gamma A r^{-1}\right)\left(2\Omega\lambda + \gamma A r\right) = 0$$

$$\lambda^4 + 2\gamma\lambda^3 + \left(\gamma^2 + \Omega_r^2\right)\lambda^2 - 2\Omega\zeta_0\gamma\lambda + \gamma^2 A^2 = 0.$$

The solutions all decay provided that $\mathrm{Re}(\lambda) < 0$ for all four roots of the quartic equation.

Assume that $\Omega_r^2 > 0$; otherwise we have orbital and hydrodynamic instability.

- In the limit of large γ (small St, small particles), the product of roots is $O(\gamma^2)$ and the sum of roots is $O(\gamma^1)$. Two roots are $O(\gamma^1)$ and two are $O(\gamma^0)$. The first scaling gives the balance $\lambda^4 + 2\gamma\lambda^3 + \gamma^2\lambda^2 \sim 0$, i.e. $\lambda \sim -\gamma$ (twice). These solutions decay. The second scaling gives the balance $\gamma^2\lambda^2 + \gamma^2 A^2 \sim 0$, i.e. $\lambda \sim \pm iA$. Expanding further, with $\lambda \sim \pm iA + c\gamma^{-1}$, gives, at $O(\gamma^1)$, $2(\pm iA)^3 + 2(\pm iA)c - 2\Omega\zeta_0(\pm iA) = 0$, i.e. $c = A^2 + \Omega\zeta_0$. These solutions decay if $\Omega\zeta_0 < -A^2$.
- In the limit of small γ (large St, large particles), the product of roots is $O(\gamma^2)$ and the sum of roots is $O(\gamma^1)$. Two roots are $O(\gamma^0)$ and two are $O(\gamma^1)$. The first scaling gives the balance $\lambda^4 + \Omega_r^2\lambda^2 \sim 0$, i.e. $\lambda \sim \pm i\Omega_r$. Expanding further, with $\lambda \sim \pm i\Omega_r + c\gamma$, gives, at $O(\gamma^1)$, $4(\pm i\Omega_r)^3 c + 2(\pm i\Omega_r)^3 + 2\Omega_r^2(\pm i\Omega_r)c - 2\Omega\zeta_0(\pm i\Omega_r) = 0$, i.e. $c = -1 - (\Omega\zeta_0/\Omega_r^2)$. These solutions decay if $\Omega\zeta_0 > -\Omega_r^2$. The second scaling gives the balance $\Omega_r^2\lambda^2 - 2\Omega\zeta_0\gamma\lambda + \gamma^2 A^2 = 0$, i.e. $\lambda \sim (\Omega\zeta_0 \pm \sqrt{(\Omega\zeta_0)^2 - \Omega_r^2 A^2})(\gamma/\Omega_r^2)$. These solutions decay if $\Omega\zeta_0 < 0$.

For all roots to decay in both limits, we require $-\Omega_r^2 < \Omega\zeta_0 < -A^2$. For a Keplerian disc this translates into anticyclonic vortices with $r > 3$. Vortices of such shapes trap particles of all sizes.

Exercise: Use the Routh–Hurwitz stability criteria (§14.1) to verify this conclusion.

Please send any comments and corrections to `gio10@cam.ac.uk`

Dynamics of Astrophysical Discs — Mathematical Tripos, Part III

Dr G. I. Ogilvie — Lent Term 2012

NOTES ON AFD AND MHD

[*These notes are intended for reference but go well beyond what is required for the course.*]

1.1. Astrophysical fluid dynamics

1.1.1. Introduction

Astrophysical fluid dynamics (AFD) is a theory relevant to the description of the interiors of stars and planets, exterior phenomena such as discs, winds and jets, and also the interstellar medium, the intergalactic medium and cosmology itself.

A fluid description is not applicable (i) in solidified regions such as the rocky cores of giant planets, and (ii) in very tenuous regions where the mean free path of the particles is not much less than the characteristic macroscopic length-scale of the system (or indeed in any system if one examines length-scales comparable to or smaller than the mean free path).

There are various flavours of AFD in common use. The basic model involves a compressible, inviscid fluid and is Newtonian (i.e. non-relativistic). The thermal physics of the fluid may be treated in different ways, either by assuming it to be isothermal or adiabatic, or by including radiation processes in varying levels of detail.

Magnetohydrodynamics (MHD) generalizes this theory by including the dynamical effects of magnetic fields. Often the fluid is assumed to be perfectly electrically conducting (*ideal MHD*).

One can also include the dynamical (rather than thermal) effects of radiation, resulting in a theory of *radiation (magneto)hydrodynamics*. Dissipative effects such as viscosity and resistivity can be included. All these theories can also be formulated in a relativistic framework.

AFD typically differs from 'laboratory' or 'engineering' fluid dynamics in the relative importance of certain effects. Effects that may be important in AFD include compressibility, gravitation, magnetic fields, radiation forces and relativistic phenomena. Effects that are usually unimportant include viscosity, surface tension and the presence of solid boundaries.

1.1.2. Basic equations

A *simple fluid* is characterized by a velocity field $\boldsymbol{u}(\boldsymbol{r},t)$ and two independent thermodynamic properties, of which the most useful are the dynamical quantities: the mass density $\rho(\boldsymbol{r},t)$ and the pressure $p(\boldsymbol{r},t)$. Other properties, such as the temperature and the viscosity, can be regarded as functions of ρ and p.

In the *Eulerian viewpoint* we consider how the properties of the fluid vary in time at a point that is fixed in space, i.e. attached to the (usually inertial) coordinate system. In the *Lagrangian viewpoint* we consider how the properties of the fluid vary at a point that moves with the fluid. The *Eulerian time-derivative* of the density, for example, is

$$\frac{\partial \rho}{\partial t},$$

while the *Lagrangian time-derivative* is

$$\frac{\mathrm{D}\rho}{\mathrm{D}t} = \frac{\partial \rho}{\partial t} + \boldsymbol{u} \cdot \nabla \rho.$$

Equation of mass conservation

The *equation of mass conservation*,

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \boldsymbol{u}) = 0$$

has the typical form of a conservation law. Note that $\rho \boldsymbol{u}$ is the *mass flux density.*

This equation can also be written in the form

$$\frac{\mathrm{D}\rho}{\mathrm{D}t} = -\rho \nabla \cdot \boldsymbol{u}.$$

In an *incompressible fluid*, fluid elements preserve their density, and so

$$\nabla \cdot \boldsymbol{u} = 0.$$

Although no fluid is strictly incompressible, this can sometimes be a good approximation for subsonic motions.

Equation of motion

The *equation of motion* is

$$\rho \frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t} = -\rho \nabla \Phi - \nabla p + \nabla \cdot \mathbf{T},$$

and derives from Newton's second law. Here Φ is the gravitational potential and $\mathbf{T}$ is the stress tensor, a symmetric tensor field of second rank. Many types of force, such as viscous, turbulent or magnetic forces, can be represented in the form $\nabla \cdot \mathbf{T}$.

Poisson's equation

The gravitational potential is generated by the mass distribution according to *Poisson's equation*,

$$\nabla^2\Phi = 4\pi G\rho,$$

where G is Newton's constant. The solution

$$\Phi(\boldsymbol{r},t) = -G\int \frac{\rho(\boldsymbol{r}',t)}{|\boldsymbol{r}'-\boldsymbol{r}|}\,\mathrm{d}^3\boldsymbol{r}'$$

will, in general, involve contributions from both inside and outside the fluid body being considered. Some astrophysical discs are *non-self-gravitating*, meaning that the internal contribution can be neglected relative to that of external masses, most importantly the central object. In this case Φ is known in advance and Poisson's equation is not coupled to the other equations.

Thermal energy equation

The *thermal energy equation* is

$$\rho T\frac{\mathrm{D}s}{\mathrm{D}t} = \mathscr{H} - \mathscr{C},$$

where T is the temperature, s the specific entropy (entropy per unit mass), $\mathscr{H}$ the heating rate per unit volume and $\mathscr{C}$ the cooling rate per unit volume. Note that $\mathscr{H}$ and $\mathscr{C}$ represent non-adiabatic processes.

In the case of a (Navier–Stokes) viscous stress,

$$\mathbf{T} = 2\mu\mathbf{S} + \mu_\mathrm{b}(\nabla\cdot\boldsymbol{u})\mathbf{I},$$

where μ is the (shear) viscosity, μ_b the bulk viscosity and

$$\mathbf{S} = \frac{1}{2}\left[\nabla\boldsymbol{u} + (\nabla\boldsymbol{u})^\mathrm{T}\right] - \frac{1}{3}(\nabla\cdot\boldsymbol{u})\mathbf{I}$$

is the traceless shear tensor. ($\mathbf{I}$ is the unit tensor. Recall that the *dynamic viscosity* μ and the *kinematic viscosity* ν are related by $\mu = \rho\nu$.) The viscous heating rate is

$$\mathscr{H} = \mathbf{T} : \nabla\boldsymbol{u} = 2\mu\mathbf{S}^2 + \mu_\mathrm{b}(\nabla\cdot\boldsymbol{u})^2.$$

Although molecular viscosity is rarely important in astrophysics, it is commonly used to parametrize other processes, especially turbulence, that to some extent mimic it.

The radiative cooling rate can be written

$$\mathscr{C} = \nabla\cdot\boldsymbol{F},$$

where $\boldsymbol{F}$ is the *radiative energy flux density.* In optically thick media the radiation is locally close to a black-body distribution and may be treated in the *diffusion approximation.* The radiative flux is then directed down the temperature gradient,

$$\boldsymbol{F} = -\frac{16\sigma T^3}{3\kappa\rho}\nabla T,$$

where σ is the Stefan–Boltzmann constant and κ the *Rosseland mean opacity.*

To relate the thermal quantities T and s to the dynamical variables ρ and p, an *equation of state* is required, together with standard thermodynamic identities. (The local thermodynamic state of a simple fluid is specified by any two of these quantities. This ignores the possible complications of variable chemical composition.) Most important in astrophysics is the case of an *ideal gas together with black-body radiation*, for which

$$p = p_{\mathrm{g}} + p_{\mathrm{r}} = \frac{k_{\mathrm{B}}\rho T}{\mu_{\mathrm{m}} m_{\mathrm{p}}} + \frac{4\sigma T^4}{3c},$$

where k_{B} is Boltzmann's constant, m_{p} the mass of the proton and c the speed of light. μ_{m} is the mean molecular weight (2.0 for molecular hydrogen, 1.0 for atomic hydrogen, 0.5 for fully ionized hydrogen and about 0.6 for ionized matter of cosmic abundances). Radiation pressure is often negligible but can be important in the centres of stars or in the innermost part of luminous discs around neutron stars and black holes.

In AFD it is usual to define three *adiabatic exponents* γ_1, γ_2 and γ_3 by

$$\begin{aligned}
\gamma_1 &= \left(\frac{\partial \ln p}{\partial \ln \rho}\right)_s, \\
\frac{\gamma_2}{\gamma_2 - 1} &= \left(\frac{\partial \ln p}{\partial \ln T}\right)_s, \\
\gamma_3 - 1 &= \left(\frac{\partial \ln T}{\partial \ln \rho}\right)_s.
\end{aligned}$$

The ratio of specific heats $\gamma = c_{\mathrm{p}}/c_{\mathrm{v}}$ and the three gamma coefficients are related by

$$\gamma_1 = \left(\frac{\gamma_2}{\gamma_2 - 1}\right)(\gamma_3 - 1) = \chi_\rho + \chi_T(\gamma_3 - 1) = \chi_\rho\gamma, \tag{1}$$

where

$$\chi_\rho = \left(\frac{\partial \ln p}{\partial \ln \rho}\right)_T, \qquad \chi_T = \left(\frac{\partial \ln p}{\partial \ln T}\right)_\rho$$

can be found from the equation of state. In the case of an ideal gas (with negligible radiation pressure), $\chi_\rho = \chi_T = 1$ and so $\gamma_1 = \gamma_2 = \gamma_3 = \gamma$.

We then have the important relation

$$\rho T \,\mathrm{d}s = \left(\frac{1}{\gamma_3 - 1}\right)\left(\mathrm{d}p - \frac{\gamma_1 p}{\rho}\mathrm{d}\rho\right), \tag{2}$$

which allows the thermal energy equation to be rewritten in terms of the dynamical variables ρ and p, i.e.

$$\left(\frac{1}{\gamma_3 - 1}\right)\left(\frac{\mathrm{D}p}{\mathrm{D}t} - \frac{\gamma_1 p}{\rho}\frac{\mathrm{D}\rho}{\mathrm{D}t}\right) = \mathbf{T} : \nabla \boldsymbol{u} - \nabla \cdot \boldsymbol{F}.$$

It is often assumed that $\gamma_1 = \gamma_2 = \gamma_3 = \gamma$.

Exercise: Derive equations (1) and (2). For the latter you will require the Maxwell relation that comes from the expression $\mathrm{d}e = T\,\mathrm{d}s - p\,\mathrm{d}(\rho^{-1})$ for the differential of the specific internal energy.

1.2. Elementary derivation of the MHD equations

Magnetohydrodynamics (or MHD) is the dynamics of an electrically conducting fluid containing a magnetic field. It is a fusion of fluid dynamics and electromagnetism.

1.2.1. The induction equation

We restrict ourselves to a non-relativistic theory in which fluid motions are slow compared to the speed of light. The electromagnetic fields $\boldsymbol{E}$ and $\boldsymbol{B}$ are governed by Maxwell's equations without the displacement current,

$$\begin{aligned}\frac{\partial \boldsymbol{B}}{\partial t} &= -\nabla \times \boldsymbol{E},\\ \nabla \cdot \boldsymbol{B} &= 0,\\ \nabla \times \boldsymbol{B} &= \mu_0 \boldsymbol{J},\end{aligned}$$

where μ_0 is the permeability of free space and $\boldsymbol{J}$ is the electric current density. The fourth Maxwell equation, involving $\nabla \cdot \boldsymbol{E}$, is not required in a non-relativistic theory.

Exercise: Show that these equations are invariant under the *Galilean transformation* to a frame of reference moving with uniform relative velocity $\boldsymbol{v}$,

$$\begin{aligned}\boldsymbol{r}' &= \boldsymbol{r} - \boldsymbol{v}t,\\ t' &= t,\\ \boldsymbol{E}' &= \boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B},\\ \boldsymbol{B}' &= \boldsymbol{B},\\ \boldsymbol{J}' &= \boldsymbol{J},\end{aligned}$$

as required by a 'non-relativistic' theory. (In fact, this is simply 'Galilean relativity'.)

Ohm's law for a static medium with electrical conductivity σ is

$$\boldsymbol{J} = \sigma \boldsymbol{E}.$$

For an electrically conducting fluid moving with velocity $\boldsymbol{u}$ this becomes

$$\boldsymbol{J} = \sigma(\boldsymbol{E} + \boldsymbol{u} \times \boldsymbol{B}),$$

or

$$\boldsymbol{E} = -\boldsymbol{u} \times \boldsymbol{B} + \frac{\boldsymbol{J}}{\sigma}.$$

From Maxwell's equations we then obtain the *induction equation*

$$\frac{\partial \boldsymbol{B}}{\partial t} = \nabla \times (\boldsymbol{u} \times \boldsymbol{B}) - \nabla \times (\eta \nabla \times \boldsymbol{B}),$$

where

$$\eta = \frac{1}{\mu_0 \sigma}$$

is the *magnetic diffusivity*. (The *resistivity* is $1/\sigma$.) Note that this is an evolutionary equation for $\boldsymbol{B}$ alone, and $\boldsymbol{E}$ and $\boldsymbol{J}$ have been eliminated. The divergence of the induction equation is

$$\frac{\partial}{\partial t}(\nabla \cdot \boldsymbol{B}) = 0,$$

so the solenoidal character of $\boldsymbol{B}$ is preserved.

1.2.2. The Lorentz force

A fluid carrying a current density $\boldsymbol{J}$ in a magnetic field $\boldsymbol{B}$ experiences a *Lorentz force*

$$\boldsymbol{F}_\mathrm{m} = \boldsymbol{J} \times \boldsymbol{B} = \frac{1}{\mu_0}(\nabla \times \boldsymbol{B}) \times \boldsymbol{B}$$

per unit volume. (The electrostatic force is negligible in the non-relativistic theory.) In Cartesian components,

$$\begin{aligned}(\mu_0 \boldsymbol{F}_\mathrm{m})_i &= \epsilon_{ijk}(\epsilon_{jlm}\partial_l B_m)B_k \\ &= (\partial_k B_i - \partial_i B_k)B_k \\ &= B_k \partial_k B_i - \partial_i(\tfrac{1}{2}B^2).\end{aligned}$$

Thus

$$\boldsymbol{F}_\mathrm{m} = \frac{1}{\mu_0}\boldsymbol{B} \cdot \nabla \boldsymbol{B} - \nabla\left(\frac{B^2}{2\mu_0}\right).$$

The first term is a *curvature force* due to a tension in the field lines. The second term is the gradient of an isotropic *magnetic pressure*

$$p_\mathrm{m} = \frac{B^2}{2\mu_0},$$

which is also equal to the energy density of the magnetic field.

Alternatively, one can write

$$\boldsymbol{F}_{\mathrm{m}} = \nabla \cdot \mathbf{M},$$

where

$$\mathbf{M} = \frac{\boldsymbol{BB}}{\mu_0} - \frac{B^2}{2\mu_0}\mathbf{I}$$

is the *Maxwell stress tensor.* If the magnetic field is locally aligned with the z-axis, then

$$\mathbf{M} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & +\frac{B^2}{\mu_0} \end{bmatrix} + \begin{bmatrix} -\frac{B^2}{2\mu_0} & 0 & 0 \\ 0 & -\frac{B^2}{2\mu_0} & 0 \\ 0 & 0 & -\frac{B^2}{2\mu_0} \end{bmatrix}.$$

The first term represents a *magnetic tension* $T_{\mathrm{m}} = B^2/\mu_0$ per unit area in the field lines. This gives rise to *Alfvén waves*, which travel parallel to the field with characteristic speed

$$v_{\mathrm{A}} = \left(\frac{T_{\mathrm{m}}}{\rho}\right)^{1/2} = \frac{B}{(\mu_0\rho)^{1/2}},$$

the *Alfvén speed.* This is often considered as a vector Alfvén velocity,

$$\boldsymbol{v}_{\mathrm{A}} = \frac{\boldsymbol{B}}{(\mu_0\rho)^{1/2}}.$$

The magnetic pressure also affects the propagation of sound waves, which become *magnetoacoustic waves.*

The combination

$$\Pi = p + \frac{B^2}{2\mu_0}$$

is often referred to as the *total pressure.* The ratio

$$\beta = \frac{p}{B^2/2\mu_0}$$

is known as the *plasma beta.*

1.2.3. Joule heating

Like viscosity, resistivity is a source of irreversibility and dissipation. (Note, however, that while viscosity is the microscopic transport coefficient for momentum, resistivity is *inversely* proportional to the microscopic transport coefficient for electrical current.) In the presence of resistivity, magnetic energy is dissipated at a rate

$$\mathscr{H}_{\mathrm{Joule}} = \frac{J^2}{\sigma} = \mu_0\eta J^2$$

per unit volume and converted into heat.

1.2.4. Summary of the MHD equations

A full set of MHD equations, including compressibility and dissipative effects, might read

$$\frac{\partial\rho}{\partial t}+\nabla\cdot(\rho\boldsymbol{u})=0,$$

$$\rho\frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t}=-\rho\nabla\Phi-\nabla p+\nabla\cdot\mathbf{T}+\frac{1}{\mu_0}(\nabla\times\boldsymbol{B})\times\boldsymbol{B},$$

$$\rho T\frac{\mathrm{D}s}{\mathrm{D}t}=\mathbf{T}:\nabla\boldsymbol{u}+\mu_0\eta J^2-\nabla\cdot\boldsymbol{F},$$

$$\nabla\cdot\boldsymbol{B}=0,$$

$$\frac{\partial\boldsymbol{B}}{\partial t}=\nabla\times(\boldsymbol{u}\times\boldsymbol{B})-\nabla\times(\eta\nabla\times\boldsymbol{B}),$$

together with constitutive relations determining the viscosity, magnetic diffusivity, opacity, equation of state, etc. Most of these equations can be written in at least one other way that may be useful in different circumstances.

These equations display the essential *nonlinearity* of MHD. When the velocity field is prescribed, an artifice known as the *kinematic approximation*, the induction equation is a relatively straightforward linear evolutionary equation for the magnetic field. However, a sufficiently strong magnetic field will modify the velocity field through its dynamical effect, the Lorentz force. This nonlinear coupling leads to a rich variety of behaviour. (Of course, the purely hydrodynamic nonlinearity of the $\boldsymbol{u}\cdot\nabla\boldsymbol{u}$ term is still present. Another, less important nonlinear effect of the magnetic field occurs through Joule heating.)

1.3. Kinematics of the magnetic field

1.3.1. Ideal MHD

For a perfect electrical conductor, $\sigma\to\infty$ and so $\eta\to 0$. This limit is known as *ideal MHD*. The induction equation becomes

$$\frac{\partial\boldsymbol{B}}{\partial t}=\nabla\times(\boldsymbol{u}\times\boldsymbol{B}).$$

This equation has a beautiful geometrical interpretation: the magnetic field lines are 'frozen in' to the fluid and can be identified with material lines. To show this, write the induction equation in the form

$$\frac{\mathrm{D}\boldsymbol{B}}{\mathrm{D}t}=\boldsymbol{B}\cdot\nabla\boldsymbol{u}-(\nabla\cdot\boldsymbol{u})\boldsymbol{B},$$

and use the equation of mass conservation,

$$\frac{\mathrm{D}\rho}{\mathrm{D}t} = -\rho\nabla\cdot\boldsymbol{u},$$

to obtain

$$\frac{\mathrm{D}}{\mathrm{D}t}\left(\frac{\boldsymbol{B}}{\rho}\right) = \left(\frac{\boldsymbol{B}}{\rho}\right)\cdot\nabla\boldsymbol{u}.$$

This is exactly the same equation satisfied by an infinitesimal *material line element* $\delta\boldsymbol{r}$ as it is stretched by the velocity gradient:

$$\frac{\mathrm{D}}{\mathrm{D}t}\delta\boldsymbol{r} = \delta\boldsymbol{u} = \delta\boldsymbol{r}\cdot\nabla\boldsymbol{u}.$$

Precisely the same equation,

$$\frac{\partial\boldsymbol{\omega}}{\partial t} = \nabla\times(\boldsymbol{u}\times\boldsymbol{\omega}),$$

is satisfied by the vorticity $\boldsymbol{\omega} = \nabla\times\boldsymbol{u}$ in barotropic (homentropic) ideal fluid dynamics in the absence of a magnetic field. However, the fact that $\boldsymbol{\omega}$ and $\boldsymbol{u}$ are directly related by the curl operation means that the analogy between vorticity dynamics and MHD is not perfect.

Another way to demonstrate the result of flux freezing is to represent the magnetic field using *Euler potentials* α and β,

$$\boldsymbol{B} = \nabla\alpha\times\nabla\beta.$$

This is sometimes called a *Clebsch representation.* By using two scalar potentials we are able to represent a three-dimensional vector field satisfying the constraint $\nabla\cdot\boldsymbol{B} = 0$. A vector potential of the form $\boldsymbol{A} = \alpha\nabla\beta + \nabla\gamma$ generates this magnetic field. The magnetic field lines are the intersections of the families of surfaces $\alpha = \text{constant}$ and $\beta = \text{constant}$.

After some algebra it can be shown that

$$\frac{\partial\boldsymbol{B}}{\partial t} - \nabla\times(\boldsymbol{u}\times\boldsymbol{B}) = \nabla\left(\frac{\mathrm{D}\alpha}{\mathrm{D}t}\right)\times\nabla\beta + \nabla\alpha\times\nabla\left(\frac{\mathrm{D}\beta}{\mathrm{D}t}\right). \tag{3}$$

The ideal induction equation is therefore satisfied if the Euler potentials are conserved following the fluid flow, i.e. if the families of surfaces $\alpha = \text{constant}$ and $\beta = \text{constant}$ are material surfaces. In this case the magnetic field lines can also be identified with material lines.

Exercise: Derive equation (3).

1.3.2. Non-ideal MHD

When $\eta > 0$ the resistivity of the fluid causes *diffusion* of the magnetic field and *dissipation* of magnetic energy. In the case of a uniform magnetic diffusivity, the induction equation becomes

$$\frac{\partial \boldsymbol{B}}{\partial t} = \nabla \times (\boldsymbol{u} \times \boldsymbol{B}) + \eta \nabla^2 \boldsymbol{B}.$$

Magnetic field lines are no longer frozen in to the fluid, but are able to slip through it. If L and U represent characteristic scales of length and velocity for the flow, the characteristic time-scales for advection and diffusion of the field are

$$T_{\text{advection}} = \frac{L}{U},$$

$$T_{\text{diffusion}} = \frac{L^2}{\eta}.$$

The relative importance of advection is measured by the *magnetic Reynolds number*

$$\mathrm{Rm} = \frac{T_{\text{diffusion}}}{T_{\text{advection}}} = \frac{LU}{\eta}.$$

When $\mathrm{Rm} \gg 1$, as is typical in astrophysics, ideal MHD is a good approximation. However, a highly conducting fluid can violate the constraints of ideal MHD by developing very small-scale structures for which Rm is not large. This happens in reconnection and in turbulence.

The magnetic diffusivity has the same dimensions as the kinematic viscosity ν. Their ratio is (one definition of) the *magnetic Prandtl number*

$$\mathrm{Pm} = \frac{\nu}{\eta}.$$

1.4. Energetics and conservation laws

1.4.1. Synthesis of the total energy equation, including dissipation

Kinetic energy:

$$\rho \frac{\mathrm{D}}{\mathrm{D}t}(\tfrac{1}{2}u^2) = -\rho \boldsymbol{u} \cdot \nabla \Phi - \boldsymbol{u} \cdot \nabla p + \boldsymbol{u} \cdot (\nabla \cdot \mathbf{T}) + \frac{1}{\mu_0} \boldsymbol{u} \cdot [(\nabla \times \boldsymbol{B}) \times \boldsymbol{B}].$$

Gravitational energy (assuming Φ to be independent of t):

$$\rho \frac{\mathrm{D}\Phi}{\mathrm{D}t} = \rho \boldsymbol{u} \cdot \nabla \Phi.$$

Thermal energy (noting $\mathrm{d}e = T\,\mathrm{d}s - p\,\mathrm{d}(\rho^{-1}) \Rightarrow \rho\,\mathrm{d}e = p\,\mathrm{d}\ln\rho + \rho T\,\mathrm{d}s$):

$$\rho \frac{\mathrm{D}e}{\mathrm{D}t} = -p \nabla \cdot \boldsymbol{u} + \mathbf{T} : \nabla \boldsymbol{u} + \mu_0 \eta J^2 - \nabla \cdot \boldsymbol{F}.$$

Sum of these three:

$$\rho\frac{\mathrm{D}}{\mathrm{D}t}(\tfrac{1}{2}u^2+\Phi+e)=\frac{1}{\mu_0}(\nabla\times\boldsymbol{B})\cdot(-\boldsymbol{u}\times\boldsymbol{B}+\eta\nabla\times\boldsymbol{B})-\nabla\cdot(p\boldsymbol{u}-\mathbf{T}\cdot\boldsymbol{u}+\boldsymbol{F})$$

or (using specific enthalpy $w=e+(p/\rho)$)

$$\frac{\partial}{\partial t}\left[\rho(\tfrac{1}{2}u^2+\Phi+e)\right]+\nabla\cdot\left[\rho\boldsymbol{u}(\tfrac{1}{2}u^2+\Phi+w)-\mathbf{T}\cdot\boldsymbol{u}+\boldsymbol{F}\right]=\frac{1}{\mu_0}(\nabla\times\boldsymbol{B})\cdot\boldsymbol{E}.$$

Magnetic energy:

$$\frac{\partial}{\partial t}\left(\frac{B^2}{2\mu_0}\right)=-\frac{1}{\mu_0}\boldsymbol{B}\cdot\nabla\times\boldsymbol{E}.$$

Total energy:

$$\frac{\partial}{\partial t}\left[\rho(\tfrac{1}{2}u^2+\Phi+e)+\frac{B^2}{2\mu_0}\right]+\nabla\cdot\left[\rho\boldsymbol{u}(\tfrac{1}{2}u^2+\Phi+w)+\frac{\boldsymbol{E}\times\boldsymbol{B}}{\mu_0}-\mathbf{T}\cdot\boldsymbol{u}+\boldsymbol{F}\right]=0.$$

Note that $(\boldsymbol{E}\times\boldsymbol{B})/\mu_0$ is the electromagnetic *Poynting flux*, while $-\mathbf{T}\cdot\boldsymbol{u}$ is the viscous energy flux.

1.4.2. Other conservation laws in ideal MHD

[*Note: The term 'ideal MHD' is usually understood to mean that all dissipative and non-adiabatic effects, including viscosity, resistivity, viscous and Joule heating, and radiative cooling, are neglected.*]

Mass:

$$\frac{\partial\rho}{\partial t}+\nabla\cdot(\rho\boldsymbol{u})=0.$$

z-component (e.g.) of momentum (not conserved in an external gravitational field):

$$\frac{\partial}{\partial t}(\rho u_z)+\nabla\cdot\left(\rho u_z\boldsymbol{u}-\frac{B_z\boldsymbol{B}}{\mu_0}+\Pi\,\boldsymbol{e}_z\right)=-\rho\frac{\partial\Phi}{\partial z}.$$

z-component (e.g.) of angular momentum (conserved only in an axisymmetric gravitational field):

$$\frac{\partial}{\partial t}(\rho r u_\phi)+\nabla\cdot\left[\rho r u_\phi\boldsymbol{u}-\frac{rB_\phi\boldsymbol{B}}{\mu_0}+r\Pi\,\boldsymbol{e}_\phi\right]=-\rho\frac{\partial\Phi}{\partial\phi}.$$

In ideal fluid dynamics there are also certain invariants with a geometrical or topological interpretation. In barotropic (homentropic) flow, for example, vorticity (or, equivalently, circulation) is conserved, while, in non-barotropic flow, potential vorticity is conserved. The Lorentz force breaks these conservation laws because the curl of the Lorentz force per unit mass does not vanish in general. However, some new topological invariants associated with the magnetic field appear.

The *magnetic helicity* in a volume V is

$$H_{\mathrm{m}} = \int_V \boldsymbol{A} \cdot \boldsymbol{B} \, \mathrm{d}V,$$

where $\boldsymbol{A}$ is the magnetic vector potential, such that $\boldsymbol{B} = \nabla \times \boldsymbol{A}$. Now

$$\begin{aligned} \frac{\partial \boldsymbol{A}}{\partial t} &= -\boldsymbol{E} - \nabla \Phi_{\mathrm{e}} \\ &= \boldsymbol{u} \times \boldsymbol{B} - \eta \nabla \times \boldsymbol{B} - \nabla \Phi_{\mathrm{e}}, \end{aligned}$$

where Φ_{e} is the electrostatic potential. This can be thought of as the 'uncurl' of the induction equation. In ideal MHD, therefore, magnetic helicity is conserved:

$$\frac{\partial}{\partial t}(\boldsymbol{A} \cdot \boldsymbol{B}) + \nabla \cdot [\Phi_{\mathrm{e}} \boldsymbol{B} + \boldsymbol{A} \times (\boldsymbol{u} \times \boldsymbol{B})] = 0.$$

However, care is needed because H_{m} is not uniquely defined unless $\boldsymbol{B} \cdot \boldsymbol{n} = 0$ on the surface S of V. Under a gauge transformation $\boldsymbol{A} \mapsto \boldsymbol{A} + \nabla \chi$, $\Phi_{\mathrm{e}} \mapsto \Phi_{\mathrm{e}} - \partial \chi / \partial t$, H_{m} changes by an amount

$$\int_V \boldsymbol{B} \cdot \nabla \chi \, \mathrm{d}V = \int_V \nabla \cdot (\chi \boldsymbol{B}) \, \mathrm{d}V = \int_S \chi \boldsymbol{B} \cdot \boldsymbol{n} \, \mathrm{d}S.$$

The *cross-helicity* in a volume V is

$$H_{\mathrm{c}} = \int_V \boldsymbol{u} \cdot \boldsymbol{B} \, \mathrm{d}V.$$

It is helpful here to write the equation of motion in ideal MHD in the form

$$\frac{\partial \boldsymbol{u}}{\partial t} + (\nabla \times \boldsymbol{u}) \times \boldsymbol{u} = -\nabla(\tfrac{1}{2}u^2 + \Phi + w) + T \nabla s + \frac{1}{\mu_0 \rho}(\nabla \times \boldsymbol{B}) \times \boldsymbol{B},$$

using the relation $\mathrm{d}w = \rho^{-1} \, \mathrm{d}p + T \, \mathrm{d}s$. Thus

$$\frac{\partial}{\partial t}(\boldsymbol{u} \cdot \boldsymbol{B}) + \nabla \cdot \left[\boldsymbol{u} \times (\boldsymbol{u} \times \boldsymbol{B}) + (\tfrac{1}{2}u^2 + w + \Phi)\boldsymbol{B}\right] = T \boldsymbol{B} \cdot \nabla s,$$

and so cross-helicity is conserved in ideal MHD in barotropic (homentropic) flow.

Note on the polytropic model and the gamma function

The gamma function is defined for $p > 0$ by

$$\Gamma(p) = \int_0^\infty x^{p-1}\, e^{-x}\, dx$$

and satisfies $\Gamma(p+1) = p\Gamma(p)$. We have $\Gamma(p) = (p-1)!$ if p is a positive integer.

Make the substitution $x = y^2$:

$$\Gamma(p) = \int_0^\infty 2y^{2p-1}\, e^{-y^2}\, dy. \tag{1}$$

Consider the integral

$$I_p = \int_{-1}^{1} (1-x^2)^p\, dx, \qquad p \geqslant 0.$$

Make the substitution $x = \cos\theta$:

$$I_p = \int_{-\pi/2}^{\pi/2} \cos^{2p+1}\theta\, d\theta.$$

Multiply both sides by a certain integral of the type (1) to make a double integral in polar coordinates:

$$\begin{aligned} I_p \int_0^r 2r^{2p+2}\, e^{-r^2}\, dr &= \int_{-\pi/2}^{\pi/2} \cos^{2p+1}\theta\, d\theta \int_0^r 2r^{2p+2}\, e^{-r^2}\, dr \\ &= \int_{-\pi/2}^{\pi/2} \int_0^\infty 2(r\cos\theta)^{2p+1}\, e^{-r^2}\, r\, dr\, d\theta. \end{aligned}$$

Transform from polar to Cartesian coordinates:

$$\begin{aligned} I_p \int_0^r 2r^{2p+2}\, e^{-r^2}\, dr &= \int_{-\infty}^\infty \int_0^\infty 2x^{2p+1}\, e^{-(x^2+y^2)}\, dx\, dy \\ &= \int_{-\infty}^\infty e^{-y^2}\, dy \int_0^\infty 2x^{2p+1}\, e^{-x^2}\, dx. \end{aligned}$$

Use equation (1) to obtain

$$I_p = \frac{\sqrt{\pi}\,\Gamma(p+1)}{\Gamma(p+\frac{3}{2})}. \tag{2}$$

The polytropic disc model is of the form

$$\tilde{\rho} = C_\rho \left(1 - \frac{\tilde{z}^2}{a^2}\right)^n, \qquad \tilde{p} = C_p \left(1 - \frac{\tilde{z}^2}{a^2}\right)^{n+1},$$

where C_ρ, C_p and a are to be determined.

To satisfy $d\tilde{p}/d\rho = -\tilde{\rho}\tilde{z}$, we require

$$2(n+1)C_p = a^2 C_\rho.$$

To satisfy the normalization conditions $\int \tilde{\rho}\, d\tilde{z} = \int \tilde{p}\, d\tilde{z} = 1$, we require

$$I_n a C_\rho = I_{n+1} a C_p = 1.$$

Thus

$$a^2 = 2(n+1)\frac{C_p}{C_\rho} = 2(n+1)\frac{I_n}{I_{n+1}} = 2(n+1)\frac{\Gamma(n+1)}{\Gamma(n+2)}\frac{\Gamma(n+\frac{5}{2})}{\Gamma(n+\frac{3}{2})} = 2(n+\tfrac{3}{2}) = 2n+3.$$

The solution is

$$C_\rho = \frac{\Gamma(n+\frac{3}{2})}{\Gamma(n+1)}\frac{1}{\sqrt{(2n+3)\pi}}, \qquad C_p = \frac{\Gamma(n+\frac{5}{2})}{\Gamma(n+2)}\frac{1}{\sqrt{(2n+3)\pi}},$$

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